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60th Annual UNSW School Mathematics Competition: Competition Problems and Solutions

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A Junior Division – Problems

Problem A1:
Two students play the following game with a box of matches. Each student takes either one, two, or three matches from the box at each turn. The student who takes the last match losses. If the number of matches in the box is N, then find the winning strategy and the winner for every N.

Problem A2:
Let a and b be two integers such that $58a = 63b$.
Prove that the number $a + b$ is a composite number.

Problem A3:
Bob writes the integers from 1 to 10 in random order. Then, Bob adds each number with the index of the place where it is written and writes them down.
Prove that two numbers in the latter sequence end with the same digit.
Note that place indexing starts with 1.

Problem A4:
Alice writes every positive integer from 1 to 10^9 in decimal form, and then she computes the sum of digits for each integer written. For each number in this new sequence, Alice computes the sum of digits again. Alice continues this process until a sequence of billion single-digit numbers is left.
In this latter sequence, what is greater: the number of 1's or the number of 2's?

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Problem A5:

As shown in the picture (a) below, two white and two black knights are positioned on a 3×3-chessboard. Find the minimum number of moves required to place the black knights in the positions of the white ones and vice versa. Solve the same problem with the initial position shown in picture (b) below.

B Senior Division – Problems

Problem B1:
Bob writes the integers from 1 to 10 in random order. Then, Bob adds each number with the index of the place where it is written and writes them down.
Prove that two numbers in the latter sequence end with the same digit.
Note that place indexing starts with 1.

Problem B2:
A flock of 60 cockatoos perches on 60 pine trees planted in a circle so that there is precisely one bird on each tree. From time to time, two cockatoos fly from one tree to the one next in opposite directions: one bird in the clockwise direction and one in the counter-clockwise direction. Is there a scenario when all birds end up on one tree?

Problem B3:
Alice starts with a number and writes the sum of its digits in her book. Of this number, she again writes the sum of its digits in her book. The game continues until a single-digit number appears in her book. What is this number if she started with 7^{59}?

Problem B4:
Prove that there is a 2022-digit composite positive integer such that it remains composite if any of its three adjacent digits are replaced by an arbitrary three-digit positive integer.

Problem B5:
Eleven individuals contributed to five different flood relief funds. Prove that there are two individuals, A and B, such that every fund to which individual A contributed is also the fund to which the individual B contributed.

Problem B6:
Let n be a positive integer. Prove that it is impossible to pave the plane by identical tiles each shaped as a convex polygon with n sides if $n \geq 7$. Consider only tilings such that the vertex of one tile meets the vertices of other tiles. Tilings, where the vertex of a tile is on the edge of another tile, are not allowed.

A Junior Division – Solutions

Solution A1.

Let N_k ($k = 0, 1, 2, \ldots$) be the number of the matches in the box after k takes and note that $N_0 = N$. If $N_k \mod 4 = 1$, then, after the move by the first student, the second student can ensure that $N_{k+2} \mod 4 = 1$. That is, in such case, the second student is a guaranteed winner. Otherwise, if $N_0 \mod 4 = 0, 2, 3$, then the first student can make $N_1 \mod 4 = 1$, and the roles of the students in the strategy above reverse.

Solution A2.

Note that

$$63(a + b) = 63a + 63b = 63a + 58a = 121a.$$

Since the numbers 63 and 121 are co-prime, we conclude that $121 = 11^2$ divides $a + b$. Hence, $a + b$ is not prime. $\square$

Solution A3.

We prove the required assertion by contradiction. Assume that all last digits in the latter sequence are different. If we add all of these digits, then we get $0+1+\cdots = 45$. On the other hand, if we add the numbers themselves, then we get $2\times(1+2+\cdots+10) = 110$. Since the last digits are not identical, we arrived at a contradiction. $\square$

Solution A4.

Answer: The number of 1's is greater by one.

The remainder of a number modulo 9 equals the remainder of the sum of its digits mod 9. Hence, the 1's will appear in place of every number from 1 to 10^9, which has the remainder $1 \mod 9$. Respectively, the 2's will appear in place of every number from 1 to 10^9, which has remainder $2 \mod 9$. The former are the elements of the arithmetic progression $1, 10, 19, \ldots$. There are 111111111 of them since

$$1 \le 9k + 1 \le 10^9$$

and, therefore,

$$0 \le k \le \frac{10^9 - 1}{9} = 111111111.$$

The latter are the elements of the arithmetic progression $2, 11, 20, \ldots$, and there are 111111110 of them since

$$1 \le 9k + 2 \le 10^9$$

and, therefore,

$$0 \le k \le \frac{10^9 - 9}{9} = 111111110.$$

Second, considering the Cartesian coordinates as shown on the diagram above, the distance from the point $F(2 - a; 1)$ to the line $x +$

B Senior Division – Solutions

Solution B1.

We prove the required assertion by contradiction. Assume that all last digits in the latter sequence are different. If we add all of these digits, then we get $0+1+\cdots = 45$. On the other hand, if we add the numbers themselves, then we get $2 \times (1+2+\cdots$

(a) if the replaced digits do not include the last digit, then the new number remains even and hence composite;

(b) if the last three digits are replaced by an even number, then the new number again remains even and hence composite;

(c) if the last three digits are replaced by an odd three-digit number, i.e., $M = N \times 10^m + k$ for k odd and $1001 \leq k \leq 1999$, then the new number M is divisible by k and hence composite.

Solution B5.

Let us index the funds S