

## UNS    School Mathematics Competition Junior Division    Problems and Solutions

Solutions by Denis Potapov<sup>1</sup>

### Problem

Every point on a line is painted using two different colours: black and white. Prove that there are always points  $A_1$ ,  $A_2$  and  $A_3$  of the same colour such that

$$A_1 A_2 = A_2 A_3 .$$

*Solution* Choose any two points of the same colour, say black,  $X$  and  $Y$ . Let now  $A$  be the centre of  $XY$ ;  $B$  be such that  $X$  is the centre of  $BY$  and  $C$  be such that  $Y$  is the centre of  $CX$

**Problem 3**  
Solve the equation

$$\sqrt{x} - \sqrt{x-1} = \sqrt{x-2} - \sqrt{x-3}.$$

*Solution* Write the equation in the form

$$\sqrt{x} + \sqrt{x-3} = \sqrt{x-2} + \sqrt{x-1}.$$

In such form, since

$$x \geq 3, \quad x-1 \geq x-2 \geq x-3, \quad x-1 \geq x-2, \quad x \geq x-1,$$

after squaring of both sides, we arrive at

$$\sqrt{x} \times \sqrt{x-3} = \sqrt{x-2} \times \sqrt{x-1}.$$

Squaring again gives

$$x \times (x-3) = (x-2) \times (x-1) \Leftrightarrow x - x^2 = x - x^2.$$

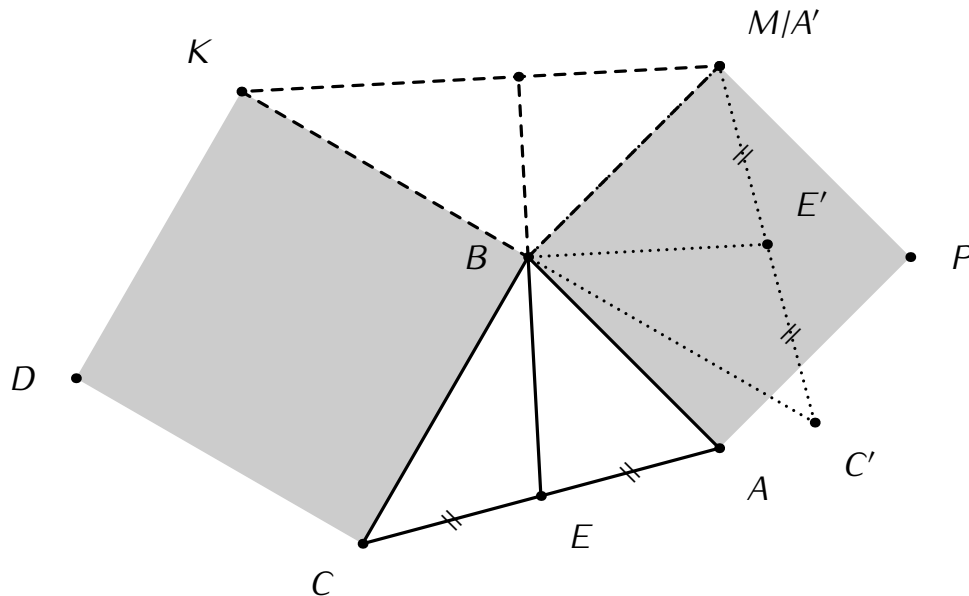
The latter solves to

$$x = 3 \quad \text{and} \quad x = 1.$$

**Problem 4**

A triangle  $ABC$  has squares  $ABMP$  and  $BCDK$  built on its outer sides. Prove that the median  $BE$  of the triangle  $ABC$  is also an altitude of the triangle  $BMK$ .

*Solution* Rotate the triangle  $ABC$  by  $90^\circ$  around vertex  $B$  as shown on the picture below. After such transformation, the median  $BE$  becomes the mid-segment of the triangle  $KMC'$ . That is, on one hand,  $BE'$  is parallel to  $KM$ , and on the other hand, it is perpendicular to the median  $BE$ .



**Problem**

**Find a five-digit number which equals 7 times the product of its digits.**

*Solution* Let

$$N = \overline{abcde}$$

be the number. We are given that

$$\overline{abcde} = 7abcde.$$

Note first that every digit  $a, b, c, d$  and  $e$  is odd. Indeed, otherwise,  $N$



# UNS School Mathematics Competition

## Senior Division Problems and Solutions

Solutions by Denis Potapov<sup>2</sup>

### Problem

A spherical planet has  $n$  satellites. Prove that there is always a point on the surface of the planet such that at most  $\lfloor n/2 \rfloor$  satellites are seen from this point.

*Proof.* Fix any two satellites, say  $S_1$  and  $S_2$ , and construct the plane through these satellites and the centre of the planet. Let  $A$  and  $B$  be the end points of the diameter of the planet perpendicular to this plane. The group of satellites visible from  $A$  does not intersect with the group of satellites visible from  $B$ . Moreover, the satellites  $S_1$  and  $S_2$  are also not visible from both point  $A$  and point  $B$ . Thus, at most

$$\lfloor n/2 \rfloor$$

are visible from either point  $A$  or point  $B$ .

### Problem

The sequence of numbers  $\{a_k\}_{k=1}^{\infty}$  is such that

$$a_k > 1$$

$$\text{and } a_{k+1} \geq a_k \frac{1}{a}, \quad k = 1, 2, \dots$$

Prove that  $a_n > \frac{1}{a}$ .

*Proof.* Since  $a_{k+1} \geq a_k \frac{1}{a}$ , it follows that

$$a_{k+1} - a_k \geq a_k \frac{1}{a} - a_k = a_k \left( \frac{1}{a} - 1 \right) = a_k \frac{1-a}{a} > 0.$$

so

$$a > \sqrt{a} > \frac{a}{2}.$$

**Problem 7**

You are given a square table filled with positive integers. On every move, you are allowed to take  $\frac{1}{2}$  from every element of a row; or to multiply every element of a column by  $\frac{1}{2}$ . Prove that there is a strategy which can reduce every element in the table to zero.

*Continuation*

$$\begin{aligned} a^2 + b^2 + c^2 &\equiv a^2 + b^2 + c^2 \pmod{4} \\ a^2 + b^2 + c^2 &\equiv a^2 + b^2 + c^2 \pmod{4} \end{aligned}$$

However  $a^2 + b^2 + c^2$  is odd, hence

$$a^2 + b^2 + c^2 \equiv 1 \pmod{4}.$$

Finally,

$$a^2 + b^2 + c^2 \leq abc$$

so

$$abc \leq abc.$$

Thus, the choices for  $a$ ,  $b$  and  $c$  are narrowed down to:

$$a, b, c$$

Let  $CD$  be the height of  $\triangle ABC$  and let  $CE$  be the diameter of the circle. Connect points  $A$  and  $E$ . The triangle  $\triangle CAE$  is right-angled since the angle  $\angle CAE$  is subtended on a diameter. Hence, the triangles

$$\triangle ACE \text{ and } \triangle CDB$$

are similar. The similarity implies that

$$\frac{CD}{CB} = \frac{AC}{CE}.$$

Solving for  $CD$  gives

$$CD = \frac{CB \times AC}{CE} = \frac{CB \times AC}{2R}.$$

By Pythagoras' Theorem applied to  $\triangle CDB$ ,

$$DB = \sqrt{CB^2 - CD^2}$$

and, by Pythagoras' Theorem on  $\triangle ADC$ ,

$$AD = \sqrt{AC^2 - CD^2}.$$

Thus,

$$AB = AD + DB.$$

### Problem

Let  $f(x)$  be a polynomial with integer coefficients and let

$$a_1, a_2, \dots, a_r$$

be integers such that for any  $n \in \mathbb{N}$ , there is an  $a_i$  such that  $f(n)$  is a multiple of  $a_i$ . Prove that there is one  $a_i$  such that  $f(n)$  is a multiple of  $a_i$  for any  $n \in \mathbb{N}$ .

*Hint:* The following theorem is known as The Chinese Remainder Theorem and it can be used in the solution of this problem without proof.

### Theorem

Let

$$q_1, q_2, \dots, q_r \in \mathbb{Z}$$

be positive pairwise coprime integers. For any integers

$$x_1, x_2, \dots, x_r \in \mathbb{Z}$$

there is an integer  $x \in \mathbb{Z}$  such that

$$x \equiv x_i \pmod{q_i}, \quad i = 1, 2, \dots, r.$$



