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2013 University of New South Wales School Mathematics Competition

Junior Division – Problems and Solutions

Problem 1

Suppose that x, y, z are non-zero integers with no common factor except 1 such that

$$x^2 + y^2 = z^2.$$

Prove that exactly one of these integers is a multiple of 5.

Solution

Let $x, y, z \in \mathbb{Z}$, $x, y, z \neq 0$

$$\gcd(x, y, z) = 1.$$

Now

x	0	1	2	3	4	mod 5
x^2	0	1	4	4	1	mod 5
$2x^2$	0	2	3	3	2	mod 5

So, if $x \equiv y \mod 5$, then

$$z^2 = x^2 + y^2 \equiv 2x^2 \mod 5$$

is possible if and only if

$$x \equiv y \equiv z \equiv 0 \mod 5.$$

This contradicts to

$$\gcd(x, y, z) = 1.$$

So $x \not\equiv y \mod 5$. Thus, we have

$$
\begin{aligned}
x^2 + y^2 &\equiv 0 + 1 \text{(in some order)} \equiv 1 && (0.1)\\
\text{or} &\equiv 0 + 4 \ldots \equiv 4 && (0.2)\\
\text{or} &\equiv 1 + 4 \ldots \equiv 0 && (0.3)\\
\text{or} &\equiv 4 + 4 \ldots \equiv 3 && (0.4)
\end{aligned}
$$

Since $z^2 \equiv 3$ is impossible, (0.4) cannot occur. So either (0.1), (0.2) or (0.3) occurs.

In (0.1) or (0.2) exactly one of $x, y \equiv 0 \mod 5$; and in (0.3) neither of $x, y \equiv 0 \mod 5$ and $z \equiv 0 \mod 5$.

Problem 2

How many integers between 100 and 100000 have exactly three identical digits in their decimal representation. For example, 11123 and 10020 are such numbers but 12121 is not.

Solution

The only 6-digit number in the range is 100000 fails the condition so the required numbers have 3, 4 or 5 digits.

Let represents the number of integers with repeating 0 and the number of integers with repeating non-zero digit.

For the integers computed by , the leading digit is non-zero, so

$$= {}_4 + {}_5;$$

where ${}_j$ is the number of j-digit integers counted by . So

$${}_j = \underbrace{9}_{A}\ \underbrace{\binom{j\quad 1}{3}}_{B}\ \underbrace{P(8; j\quad 4)}_{C};$$

where A stands for the choice for the leading digit; B stands for the choice of the places for the repeating zero; and C represents filling the remaining j 4 positions with other digits with no repetitions. So, we compute

$$\begin{aligned} &= {}_4 + {}_5 \\ &= 9 \quad 1 \quad 1 + 9 \quad \binom{4}{3} \quad 8 \\ &= 9 + 9 \quad 4 \quad 8 \\ &= 9 \quad 33 = 297. \end{aligned}$$

We split the integers counted by into three subgroups:

$$= {}_3 + {}_4 + {}_5;$$

where ${}_j$ is the number of j-digit integers within .

Within integers counted by , repeating digit may or may not include leading digit, so we compute

$${}_j = \underbrace{9}_{A}\ \underbrace{1}_{B}\ \underbrace{\binom{j\quad 1}{2}}_{C}\ \underbrace{P(9; j\quad 3)}\ 8\ ; j\{z = \quad 9$$

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where A chooses the non-zero repeating digit; B puts that digit into the leading position; C chooses two other positions for the remaining repeating digits; D fills the rest with no repetition; B^0 chooses another non-zero digit for the leading position; C^0 chooses positions for the repeating digit; D^0 fills the rest with no repetition.

Skipping some elementary computations, we conclude that

$$= 6516 \text{ and } \quad + \quad = 6813:$$

Problem 3

1. Find the number which is divisible by 2 and 9 and which has 14 divisors (including 1 and the number itself).
2. Show that there are multiple solutions in the case when the number has 15 divisors.

Solution

Let

$$n = 2^{\ 1}3^{\ 2}p_3^{\ 3}$$

or

$$_1 + 1 = 5 \text{ and } \;_2 + 2 = 3.$$

So

$$_1 = 2, \;\; _2 = 4 \text{ and } n = 2$$

1. Prove that there is a strategy such that the left hand side of the balance is lower after the first move; the right hand side of the balance is lower after

2. put all odd indexed weights on one side of the balance and all even indexed weights on the other side such that the heaviest weight is on the left hand side, if the last letter of the word is ’L’; or the right hand side if the letter is ’R’;
3. on every move, you remove the last letter from the word and a weight from the balance as follows:

wordhage3.

Senior Division – Problems and Solutions

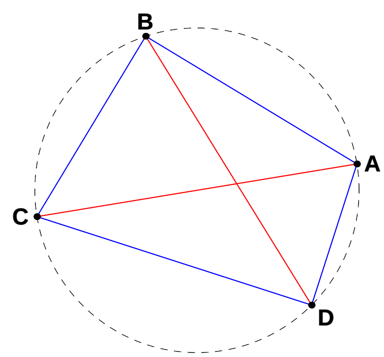
Problem 1

In this problem you may assume the following result.

Ptolemy's Theorem:

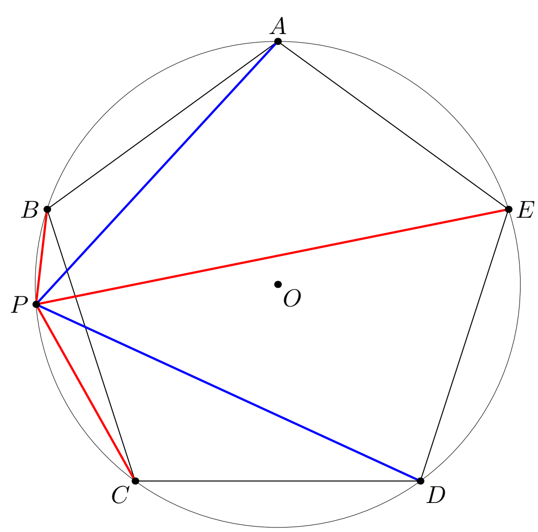
If $ABCD$ is a cyclic quadrilateral, that is a quadrilateral inscribed in a circle, then

$$BD \quad AC = AD \quad BC + AB \quad DC:$$



Suppose $ABCDE$ is a regular pentagon inscribed in a circle. Let P be any point on the arc BC. Prove that

$$PA + PD = PB + PC + PE:$$



Solution

Since we have to prove

$$PA + PD = PB + PC + PE$$

we may assume (by applying a dilation) that

$$AB = BC = CD = DE = EA = 1.$$

The five diagonals of the regular pentagon have the same length so assume that

$$AC = BD = CE = DA = EB = d.$$

With six points on the circle there are

$$\binom{6}{4} = 15$$

quadruples of points to which Ptolemy's Theorem can be applied. Fortunately the 5 sets corresponding to the points of that pentagon give the same equation. Consider the quadrilateral $ABCD$:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC \quad d^2 = 1 + d \tag{0.5}$$

The exact value of d,

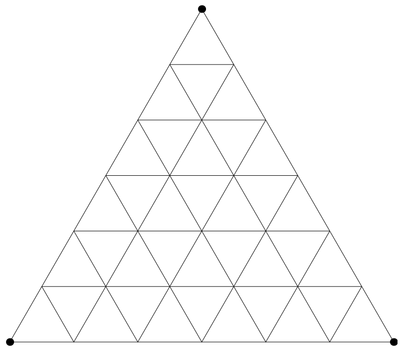
$$d = \frac{1}{2}(1 + \sqrt{5}),$$

is not needed.

Apply Ptolemy's Theorem to $PBAE$:

Problem 2

1. The plan of an art museum is an equilateral triangle consisting of 36 triangular exhibition halls (see the diagram). Each hall has passages into all adjacent halls. Prove that you can visit at most 31 halls if you plan to enter each hall at most once.



2. Subject to the condition that each hall is visited at most once, find the largest number of halls that can be visited in the case that the museum is an equilateral triangle and that it has k^2 exhibition halls, where k is an integer.

Solution

Starting from the top and working systematically along each row it appears that one can only visit 36 5 = 31 rooms.

Let a_k be the number of rooms which can be visited, then apparently

k	=	1	2	3	4	5	6
a_k	=	1	3	7	13	21	31

and for k, in general,

$$a_k = k^2 \quad (k \quad 1) = k^2 \quad k + 1:$$

Indeed, the method described above shows that

$$a_6 \quad 31 \text{ and } a_k \quad k^2 \quad k + 1;$$

in general. To see that 31 is the most possible, colour the up triangles red and the down triangles white.

Each path must go . . . RWRW. . . There are 21 red triangles and 15 white triangles so a maximal path can contain no more than 16 red and 15 white triangles, so a_6 31. Thus,

$$a_6 = 31.$$

If there are k^2 exhibition halls, then the number of red halls is

$$1 + 2 + \ldots + k = \frac{1}{2}k(k + 1)$$

and the number of white halls is

$$1 + 2 + \ldots + (k \quad 1) = \frac{1}{2}k(k \quad 1).$$

So the maximal possible path visits

$$\frac{1}{2}k(k \quad 1) + \frac{1}{2}k(k \quad 1) + 1 = k^2 \quad k + 1$$

halls.

Problem 3

Given an integer n, find the largest integer k such that 3^k is a factor of

$$2^{3^n} + 1.$$

Give an argument to justify your answer.

Solution

For a few initial values of n, we have

$$n = 0: \quad 2^{3^0} + 1 = 3;$$

$$n = 1: \quad 2^3 + 1 = 8 + 1 = 9;$$

$$n = 2: \quad 2^{3^2} + 1 = 512 + 1 = 27 \quad 19.$$

Thus, we hypothesise

$$k = n + 1$$

and a proof by induction is available. Assume

$$2^{3^n} + 1 = 3^{n+1} \quad A_n$$

with n 2 and $(3; A_n) = 1$. Then,

2)= 1

Solution

The required number of locks is

$$n = \binom{7}{3} = 35.$$

Assume first that we have some amount of locks installed and some distribution of keys between the people such that the conditions of the problem are met. That is, every group of three people cannot open the vault and any group of four can open the vault.

In this case, for every group of three people, there is a unique lock such that this group cannot open this lock, i.e., none in the group has a key to the lock. Indeed, if this was not so, that is, if there was two different groups of three people, say A and B, and a lock, say L, such that both A and B cannot open the lock L, then, by joining a person from B to the group A, we end up with a group of four people which cannot open the lock L. This is in contradiction with the assumption above. Hence, every group of three people has at least one unique lock associated with it. There are $\binom{7}{3}$ groups of 3 people and so the number of locks is at least

$$\binom{7}{3} = 35.$$

This number of locks is sufficient to ensure the requirement of the problem. Indeed, given that that we install $\binom{7}{3}$ locks, we associate uniquely a lock with a group of three people and we give keys to this lock to everyone but the associated group of three.