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2012 University of New South Wales School Mathematics Competition

Junior Division – Problems and Solutions

Problem 1 e n n te nested rad ca

$$
c = \sqrt{1 + 2\sqrt{1 + 2\
$$

convergies. Find c

Solution 1 Note t at \mathbf{f} $c =$ v9 63i797(8081 T(7 Td

More Eenera y t suseful to consider the pattern of numbers in mod

OW								
OW					$1 \quad 1 \quad 1$			
OW				1 0 1 0 1				
OW			1 1 0 1 0 1 1					
OW			1 0 0 0 1 0 0 0 1					
OW			1 1 1 0 1 1 1 0 1 1 1					
OW	1 0 1 0 0 0 1 0 0 0 1 0 1							

Loo **ge** at the rst four entries in each row start **ge** at low we see that the pattern repeats by construct on after ow wt a neac row.

uppose t at $b_{\mathbf{k},3} = \frac{1}{2}$ $\frac{1}{2}$ (+1) t en

b

$$
b_{k+1,3} = b_{k,1} + b_{k,2} + b_{k,3}
$$

= 1 + 1

stralight ine direction up a path of constant gradient. In ey walk under a tall billboard after twenty etres and continue walks up the path for another ventures at which \mathbf{r} po nt t ey turn around and not ce t at t e top of t e b board as Ens or zontally with the top of the building. They continue a $\log E$ up the path a further ten metres where they turn around ϵ and notice that the top half of the building E is now visible above teb boarde geelt of tebuilge is ucgereater tan te geelt of teb

cenar o

ey return from E to D f the roll of the dice adds up to a number divisible by t ree ot erwise they over on from E to A

Solution 6

Let P_X denote the long-term probability that the sales person stays overnight in town X and et p_{YZ} denote the transition probability that the sales person Eloes from Y to Z en $p_{AB} = 1$, $p_{BC} = 1$, $p_{CD} = 1$, $p_{DE} = 1$, $p_{ED} = p$, $p_{EA} = 1 - p$ w ere $0 \le p \le 1$

e a so ave $P_A = P_E \times p_{EA}$ $P_B = P_A \times p_{AB}$ $P_C = P_B \times p_{BC}$ $P_D = P_C \times p_{CD} + P_E \times p_{CD}$ p_{ED} $P_E = P_D \times p_{DE}$ us $P_A = P_E \times (1 - p)$ $P_B = P_A$ $P_C = P_A$ $P_D = P_A + P_E \times p$ $P_{\mathsf{E}} = P_{\mathsf{D}}$ But $P_{\mathsf{A}} + P_{\mathsf{B}} + P_{\mathsf{C}} + P_{\mathsf{D}} + P_{\mathsf{E}} = 1$ so that $4P_{\mathsf{A}} + (p+1)P_{\mathsf{E}} = 1$. Finally elimination P_{A} fro $P_{\mathsf{A}} = P_{\mathsf{E}} \times (1-p)$ and $4P_{\mathsf{A}} + (p+1)P_{\mathsf{E}} = 1$, we have $4P_{\mathsf{E}} \times (1-p) + (p+1)P_{\mathsf{E}} = 1$ and t en $P_{\mathsf{E}} = \frac{1}{\sqrt{2}}$ [−]3^p

In cenar o t e su of t e d ce s d v s b e by f t e su s one of $2, 4, 6, 8, 10, 12$ so t at $p = \frac{1+3+1}{3} = \frac{1}{2}$ $\frac{1}{2}$ and $P_{\mathsf{E}} = \frac{2}{\ell}$

 $\frac{2}{\pi}$ and $\frac{2}{\pi}$ and $\frac{2}{\pi}$ $\frac{2}{\pi}$ and $\frac{2}{\pi}$ $\frac{2}{\pi}$ $\frac{2}{\pi}$ $\frac{2}{\pi}$ of the dice is divisible by fittesus one of 3, 6, 9, 12 so t at t e probability s $p = \frac{2+4+1}{3} = \frac{1}{3}$ $\frac{1}{3}$ and $P_{\sf E} = \frac{1}{4}$ 4

Senior Division – Problems and Solutions

Problem 1

A trave **g** is sales person tours towns A, B, C, D, E and stays overnight in one of the towns. If they stay overneed that in town A then the next next next they stay in town B. If t ey stay overnet the town B then the next next they stay in town C. If they stay overn_{ight} in town C t en t e next n_{ight} they stay in town D. If they stay overnight in town D t en t e next not they stay in town E. If they stay overnote that in town E they ro two fair dice to determine whether they will return to D for the next next return over on to town A for the next next next eighther extra next nue their tour ether from D to E or from A to B etc. at steorupter probability of nd \mathbf{E} them itown E on any E ven nE t n each of the scenarios below:

cenar o

ey return from E to D f the roll of the dice adds up to a number divisible by two otherwise they over on from E to A

cenar o

ey return from E to D f the roll of the dice adds up to a number divisible by t ree ot erwise they over on from E to A

Solution 1

ee outon nte Junor Dvs f d $\begin{array}{ccc} \text{d} \text{ to} & \text{r} & \text{d} & \text{f} \end{array}$ Now cons der

$$
S = \sum_{j=0}^{\infty} \frac{j}{n^j}
$$

\n
$$
= \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \frac{5}{n} + \cdots
$$

\n
$$
= \frac{1}{n} + \frac{1}{n} \left(\frac{2}{n} + \frac{3}{n^2} + \frac{4}{n^3} + \frac{5}{n^4} + \cdots \right)
$$

\n
$$
= \frac{1}{n} + \frac{1}{n} \left(\frac{1+1}{n} + \frac{1+2}{n^2} + \frac{1+3}{n^3} + \frac{1+4}{n^4} + \cdots \right)
$$

\n
$$
= \frac{1}{n} + \frac{1}{n} \left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots \right) + \frac{1}{n} \left(\frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \cdots \right)
$$

\n
$$
= \frac{1}{n} + \frac{1}{n} \left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots \right) + \frac{1}{n} S
$$

 $S = E t$ e we nown result for $t \in E$ eo etric series

$$
\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \dots = \frac{1}{n-1}
$$

we now ave

$$
S = \frac{1}{n} + \frac{1}{n} \left(\frac{1}{n-1} \right) + \frac{1}{n} S
$$

Solution 6

e are given $(x) = (x + (x))$ for a x so that if we replace x by $x + (x)$ we have $(x + (x)) = (x + (x) + (x + (x))$ and f we now use the equality $(x + (x)) = (x + (x)) + (x + (x))$ (x) we obtain $(x) = (x + 2(x))$. Continuing in the set of the ave $(x) = (x + 1)(x - 1)$ $n(x)$ for a x and a ntellers n

e consider a proof by contradiction to show that (x) is a constant function. uppose that (x) is not a constant function. When the cost of European type and assu et ere exists $z \in (x, x + (x))$ such that $(x) < (z) < 2(x)$ and furthermore $(z) = (z + n)(z)$ for a ntellers n

C early there exists a straight line ℓ that separates the point $(z, (z))$ on the graph from points $(x, (x))$ and $(x + (x), (x))$. Without loss of Eeneral ty we suppose that t e stra \mathbf{E} , t ne ℓ \mathbf{E} ven by $y = -\frac{1}{n}$ $\frac{1}{m}x + c$ w ere m s a positive integer. It follows from t e continuity of (x) that there are at least two points $(a, (a))$ and $(b, (b))$ with $a \neq b$ t at eon the graph and the straight ine. It is shown schematically in the graph schematically in the figure.

us we ave $c = a + m(a)$ and $c = b + m(b)$ so that $(c) = (a + m(a))$ and $(c) = (b + m)(b)$. But $(a + m)(a) = (a)$ and $(b + m)(b) = (b)$ so t at $(c) = (a) = (b)$