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2012 University of New South Wales School Mathematics Competition

Junior Division – Problems and Solutions

Problem 1

e n n te nested rad ca

$$c = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{\dots}}}}}$$

conve?es F nd c

Solution 1

Note t at f

$$c =$$

More Eenera y t s usefu to cons der t e pattern of nu bers n od

ow							1						
ow						1	1	1					
ow					**1**	**0**	**1**	**0**	1				
ow				**1**	**1**	**0**	**1**	0	1	1			
ow			**1**	**0**	**0**	**0**	1	0	0	0	1		
ow		**1**	**1**	**1**	**0**	1	1	1	0	1	1	1	
ow	**1**	**0**	**1**	**0**	0	0	1	0	0	0	1	0	1

Loo E at t e rst four entr es n eac row start E at ow we see t at t e pattern repeats by construct on after ow w t a n eac row

uppose t at $b_{k,3} = \frac{1}{2}(\quad+1)$ t en

$$b_{k+1,3} = b_{k,1} + b_{k,2} + b_{k,3}$$
$$= 1 + \quad + \ 1$$

straE t ne d rect on up a pat of constantErad ent ey wa under a ta b board after twenty etres and cont nue wa nE up t e pat for anot er ve etres at w c po nt t ey turn around and not ce t at t e top of t e b board a Ens or zonta y w t t e top of t e bu d nE ey cont nue a onE up t e pat a furt er ten etres w ere t ey turn around Ea n and not ce t at t e top a f of t e bu d nE s now v s b e above t e b board e eE t of t e bu d nE s uc Ereater t an t e eE t of t e b cenar o

ey return fro E to D f t e ro of t e d ce adds up to a nu ber d v s b e by t ree ot erw se t ey ove on fro E to A

Solution 6

Let P_X denote t e o E ter probab ty t at t e sa es person stays overn E t n town X and et p_{YZ} denote t e trans t on probab ty t at t e sa es person Eoes fro Y to Z en $p_{AB} = 1, p_{BC} = 1, p_{CD} = 1, p_{DE} = 1, p_{ED} = p, p_{EA} = 1 - p$ w ere $0 \leq p \leq 1$

e a so ave $P_A = P_E \times p_{EA}$ $P_B = P_A \times p_{AB}$ $P_C = P_B \times p_{BC}$ $P_D = P_C \times p_{CD} + P_E \times p_{ED}$ $P_E = P_D \times p_{DE}$ us $P_A = P_E \times (1 - p)$ $P_B = P_A$ $P_C = P_A$ $P_D = P_A + P_E \times p$ $P_E = P_D$ But $P_A + P_B + P_C + P_D + P_E = 1$ so t at $4P_A + (p+1)P_E = 1$ F na y e nat E P_A fro $P_A = P_E \times (1-p)$ and $4P_A + (p+1)P_E = 1$ we ave $4P_E \times (1-p) + (p+1)P_E = 1$ and t en $P_E = \frac{1}{-3p}$

In cenar o t e su of t e d ce s d v s b e by f t e su s one of $2, 4, 6, 8, 10, 12$ so t at $p = \frac{1+3+ \; + \; +3+1}{3} = \frac{1}{2}$ and $P_E = \frac{2}{}$

In cenar o t e su of t e d ce s d v s b e by f t e su s one of $3, 6, 9, 12$ so t at t e probab ty s $p = \frac{2+ \; +4+1}{3} = \frac{1}{3}$ and $P_E = \frac{1}{4}$

Senior Division – Problems and Solutions

Problem 1

A trave sa es person tours towns A, B, C, D, E and stays overn t n one of t e towns If t ey stay overn t n town A t en t e next n t t ey stay n town B If t ey stay overn t n town B t en t e next n t t ey stay n town C If t ey stay overn t n town C t en t e next n t t ey stay n town D If t ey stay overn t n town D t en t e next n t t ey stay n town E If t ey stay overn t n town E t ey ro two fa r d ce to deter ne w et er t ey w return to D for t e next n t or ove on to town A for t e next n t t ey t en cont nue t e r tour e t er fro D to E or fro A to B etc at s t e o ter probab ty of nd t e n town E on any ven n t n eac of t e scenar os be ow

cenar o

ey return fro E to D f t e ro of t e d ce adds up to a nu ber d v s b e by two ot erw se t ey ove on fro E to A

cenar o

ey return fro E to D f t e ro of t e d ce adds up to a nu ber d v s b e by t ree ot erw se t ey ove on fro E to A

Solution 1

ee o ut on n t e Jun or D v s f d to r d f

Now cons der

$$
\begin{aligned}
S &= \sum_{j=0}^{\infty} \frac{j}{n^{j}} \\
&= \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \frac{5}{n} + \cdots \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{2}{n} + \frac{3}{n^2} + \frac{4}{n^3} + \frac{5}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1+1}{n} + \frac{1+2}{n^2} + \frac{1+3}{n^3} + \frac{1+4}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots\right) + \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3} + \frac{4}{n^4} + \cdots\right) \\
&= \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots\right) + \frac{1}{n}S
\end{aligned}
$$

s nE t e we nown resu t for t eEeo etr c ser es

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots = \frac{1}{n-1}$$

we now ave

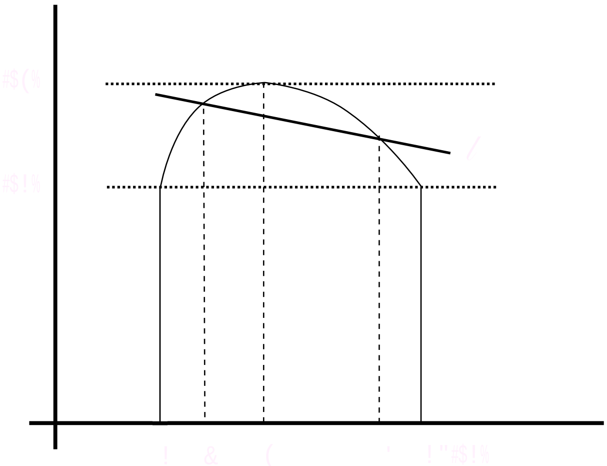
$$S = \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n-1}\right) + \frac{1}{n}S$$

Solution 6

e are g ven $(x) = (x + (x))$ for a x so t at f we rep ace x by $x + (x)$ we ave $(x + (x)) = (x + (x) + (x + (x))$ and f we now use t e equa ty $(x + (x)) = (x)$ we obta n $(x) = (x + 2 (x))$ Cont nu g n t s fas on we ave $(x) = (x + n (x))$ for a x and a nt gers n

e cons der a proof by contrad ct on to s ow t at (x) s a constant funct on uppose t at (x) s not a constant funct on t out oss of g enera ty we ay as su e t ere ex sts $z \in (x, x + (x))$ suc t at $(x) < (z) < 2 (x)$ and furt er ore $(z) = (z + n (z))$ for a nt gers n

C ear y t ere ex sts a stra g t ne ℓ t at separates t e po nt $(z, (z))$ on t e g rap fro po nts $(x, (x))$ and $(x + (x), (x))$ t out oss of g enera ty we suppose t at t e stra g t ne ℓ g ven by $y = -\frac{1}{m}x + c$ w ere m s a pos t ve nt g er It fo ows fro t e cont nu ty of (x) t at t ere are at east two po nts $(a, (a))$ and $(b, (b))$ w t $a \neq b$ t at e on t e g rap and t e stra g t ne s s s own sc e at ca y n t e g ure



us we ave $c = a + m (a)$ and $c = b + m (b)$ so t at $(c) = (a + m (a))$ and $(c) = (b + m (b))$ But $(a + m (a)) = (a)$ and $(b + m (b)) = (b)$ so t at $(c) = (a) = (b)$