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2009 University of New South Wales School Mathematics Competition

Junior Division – Problems and Solutions

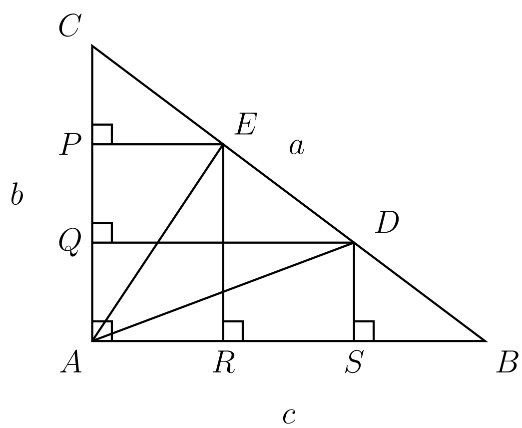
Problem 1

Let ABC be a right triangle with right angled at A. Let D, E be points on BC with $BD = DE = EC$. Prove that $AD^2 + AE^2 = \frac{5}{9}BC^2$.

Solution 1(a)

Let points R, S be on AB so that ER and DS are perpendicular to AB.

Let $AB = c, AC = b$ and $BC = a$.



So $PE \parallel QD \parallel AB$ and $DS \parallel ER \parallel AC$.

Hence since $EC = ED = DB$, then by the Equal Intercepts between Parallel Lines Theorem, we have

$$CP = PQ = QA = \frac{b}{3} \quad \text{and} \quad AR = RS = SB = \frac{c}{3}\,.$$

Alternatively, $\triangle PCE \,|||\, \triangle QCD \,|||\, \triangle ACB$ and $\triangle SDB \,|||\, \triangle REB \,|||\, \triangle ACB$, using equal right angles and corresponding angles in parallel lines, which gives the same result.

Now by Pythagoras' Theorem,

$$AD^2 = AS^2 + DS^2 = AS^2 + AQ^2 \quad \textbf{as } DS = AQ$$
$$= \left(\frac{2c}{3}\right)^2 + \left(\frac{b}{3}\right)^2 = \frac{4c^2}{} + \frac{b^2}{}$$
$$AE^2 = AR^2 + RE^2 = AR^2 + AP^2 \quad \textbf{as } RE = AP$$
$$=$$

but $AB^2 + AC^2 = BC^2 = \ x^2$, so

$$\therefore \quad (AD^2 + AE^2) + \ x^2 = 2(AD^2 + AE^2) + 4x^2$$

$$\text{so} \quad AD^2 + AE^2 = 5x^2 = \frac{5}{}BC^2.$$

Problem 2

A box contains 100 red marbles, 80 green marbles, 60 blue marbles and 40 yellow marbles.

1. **What is the smallest number of marbles, which if selected at random, will be guaranteed to contain at least ten pairs of marbles?**
2. **What is the smallest number of marbles, which if selected at random, will be guaranteed to contain at least ten triplets of marbles?**

For $n \equiv 0 \pmod 3$**, this has minimum value**

$$\frac{n}{3} - \left(\frac{2+2+2+0}{3}\right) = \frac{n}{3} - 2$$

which is ≥ 10 **iff** $n \geq 36$**.**

For $n \equiv 1 \pmod 3$**, the number of triplets has minimum value**

$$\frac{n}{3} - \left(\frac{2+2+2+1}{3}\right) = \frac{n}{3} - \frac{7}{3}$$

which is ≥ 10 **iff** $n \geq 37$**.**

For $n \equiv 2 \pmod 3$**, the number of triplets has minimum value**

$$\frac{n}{3} - \left(\frac{2+2+2+2}{3}\right) = \frac{n}{3} - \frac{8}{3}$$

which is ≥ 10 **iff** $n \geq 38$**.**

Hence the minimum no. of marbles required to guarantee 10 **triplets is** 36**.**

Problem 3

A bowl in the shape of a conical frustum is placed out in the rain at the start of a downpour. At the end of the downpour the water level

Let R_1 be the radius of the top of the bowl, R_2 be the radius of the top of the water level and r be the radius of the bottom of the bowl, and let $p = OA$.

Let V be the volume of water in the bowl.

We are given that $OB = BC = r$ and $R_1 = 2r$.

Since $\triangle AOF \; ||| \; \triangle ABE \; ||| \; \triangle ACD$,

$$\frac{p}{r} = \frac{p+2r}{R_1} = \frac{p+2r}{2r} \quad \text{so} \quad p = \frac{p}{2} + r$$

$$\therefore \quad p = 2r$$

$$\therefore \quad \frac{R_2}{r+p} = \frac{r}{p} \quad \text{so} \quad \frac{R_2}{r+2r} = \frac{1}{2}$$

$$\therefore \quad R_2 = \frac{3r}{2}.$$

$$\begin{aligned}
\text{Hence} \quad V &= \frac{1}{3}\pi R_2^2(p+r) - \frac{1}{3}\pi r^2 p \\
&= \frac{\pi}{3}\left(\frac{3r}{2}\right)^2 \cdot 3r - \frac{\pi}{3} \cdot r^2 \cdot 2r \\
&= \frac{\pi r^3}{3}\left(\frac{27}{4} - 2\right) = \frac{\pi r^3}{3} \cdot \frac{1}{4} \\
&= \frac{1 \;\; \pi r^3}{12}.
\end{aligned}$$

Rainfall is measured as the volume of water per unit area on which the water fell (assumed perpendicularly) over some time period. The water in the bowl fell on the circular cross-section at the top of the bowl, which has area $\pi R_1^2 = \pi(2r)^2 = 4\pi r^2$, hence the rainfall is

$$\frac{1 \;\; \pi r^3}{12} \cdot \frac{1}{4\pi r^2} = \frac{1 \;\; r}{48}\,.$$

Problem 4

There are many ways to substitute digits for letters in the expression

$$CAT \quad + \quad EMU \quad = \quad LION,$$

with different letters representing different digits, to obtain a valid equation. Prove that in every solution, $LION$ is a multiple of . Is there any higher number than   for which this is still true?

Solution 4

We wish to show that

$$LION \equiv 0 \pmod{\ }.$$

Without loss of generality we write

$$\begin{aligned}
CAT &\equiv a \quad \text{mod} \\
EMU &\equiv b \quad \text{mod} \\
LION &\equiv c \pmod{\ },
\end{aligned}$$

and using the digit sum rule

$$\begin{aligned} C+A+T &\equiv a \pmod{\ } \\ E+M+U &\equiv b \pmod{\ } \\ L+I+O+N &\equiv c \pmod{\ }. \end{aligned}$$

(The digit sum rule can be seen by expanding

$$CAT = 100C + 10A + T = \quad C + C + \quad A + A + T$$

so that $CAT \pmod{\ } \equiv C + A + T \pmod{\ }$ **etc.)**

Using the given equation

$$CAT + EMU = LION$$

we have

$$a + b \equiv c \pmod{\ }.$$

Now note that there are ten different letters in total so that

$$\begin{aligned} C+A+T+E+M+U+L+I+O+N &= 0+1+2+3+\ \ +\ \ +\ \ +\ \ +\ \ + \\ &= \quad = \ \cdot\ , \end{aligned}$$

and hence

$$a + b + c \equiv 0 \pmod{\ }.$$

Combining the results, $a + b \equiv c \pmod{\ }$ **and** $a+b+c \equiv 0 \pmod{\ }$**, we deduce that** $2c \equiv 0 \pmod{\ }$**, and hence** $c \equiv 0 \pmod{\ }$ **as required.**

For the second part note that $108\ \ = 324+765$ **and** $10\ 8 = 346+752$ **are both** $LION$**s and their greatest common divisor is 9.**

Problem 5

Let n **be a positive integer and let** $S_n = 1^n + 2^n + 3^n + 4^n$**.**

1. **Find** S_1, S_2, S_3, S_4**.**
2. **Show that** n^5 **has the same last digit (units digit) as** n **does.**
3. **Prove that 10 is a factor of** S_n **unless** n **is a multiple of 4.**

Solution 5

1. $S_1 = 10, S_2 = 30, S_3 = 100, S_4 = 354$
2. **One can check** $n^5 \equiv n \pmod{10}$ **for** $n = 0, 1, 2, 3, \cdots,$ **In fact we need only check this for** $n = 0, 1, 2, 3, 4, 5$ **as** $6 \equiv -2$ **etc. and we have odd powers here.**

   Hence this is true for all integers as any integer n **is** $\equiv 0, 1, 2, 3 \cdots, \pmod{10}$ **and if** $n \equiv m \pmod{10}$ **then** $n^p \equiv m^p \pmod{10}$ **for any positive integer** p**, OR**

write $n = 10k + r$ with k a non-negative integer and $r = 0, 1, 2, 3, \cdots 8$ or (Division Algorithm or Quotient-Remainder Theorem), then by the Binomial Theorem,

$$\begin{aligned} n^5 &= (10k + r)^5 \\ &= (10k)^5 + \quad \cdot (10k)^4 r + 10 \cdot (10k)^3 r^2 + 10 \cdot (10k)^2 r^3 + \quad \cdot (10k) r^4 + r^5 \\ &\equiv r^5 \equiv r \equiv n \pmod{10}. \end{aligned}$$

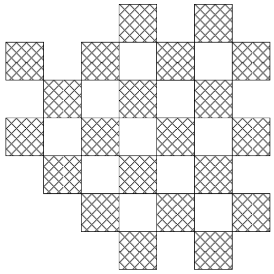
3. **For all integers n, and all non-negative integers k and positive integers r (and for $r = 0$ if $k > 0$), we have**

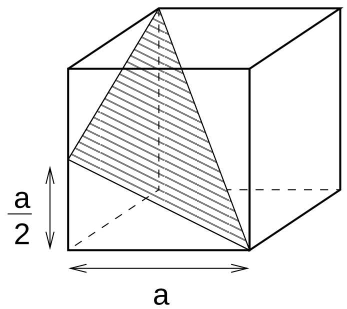
$$n^4$$

2. **The quadrophage jumps on to a random square of a large checkerboard and eventually finds that it has eaten the whole board. On which squares might it have started?**

Solution 6

1. **The case in which the quadrophage first moves down and then to the right is shown in the figure on page 38. Note that the quadrophage has no choice until it**

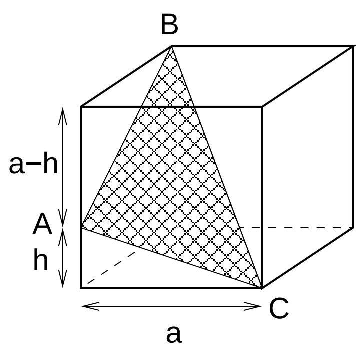




1. **It is easy to see that the given triangle has the smallest perimiter. Flatten out the faces containing the edges of the triangle, the resultant straight line is the shortest distance between the two points connecting the space diagonal. Some straightforward applications of Pythagoras's Theorem then yield the result**

$$\ell = 2\sqrt{a^2 + \left(\frac{a}{2}\right)^2} + \sqrt{3}a = (\sqrt{3} + \sqrt{5})a.$$

2. **Consider the geometry shown with the vertices of the triangle labelled A, B, C.**



Define the semi-perimeter $s = \frac{AB+BC+CA}{2}$ and the area from Heron's formula

$$\textbf{Area} \quad = \sqrt{s(s-AB)(s-BC)(s-CA)}.$$

Use Pythagoras's Theorem to write

$$\begin{aligned} AB &= \sqrt{a^2 + (a-h)^2}, \\ BC &= \sqrt{3}a, \\ CA &= \sqrt{a^2 + h^2}, \end{aligned}$$

and then after a little algebra

$$\textbf{Area} \quad = \frac{a}{\sqrt{2}}\sqrt{a^2 + h^2 - ah}.$$

Note that $\dfrac{d(\textbf{Area})}{dh} = 0$ when $h = \dfrac{a}{2}$ and $\dfrac{d^2(\textbf{Area})}{dh^2} > 0$ when $h = \dfrac{a}{2}$ so that the minimum area is Area $= \frac{\sqrt{6}}{4}a^2$.

An alternate method is to consider the minimum distance from the vertex on the edge of the cube to the space diagonal. This minimum distance is $s = \frac{a}{\sqrt{2}}$ from which the mimimum area Area $= \frac{1}{2}\frac{a}{\sqrt{2}}\sqrt{3}a = \frac{\sqrt{6}}{4}a^2$ follows.

Problem 3

Suppose that $f(0) = 1$ and $a > 0$ is a fixed real number. Show that $f(x) = 1$ is the only continuous function $f \cdot \mathbb{R} \to \mathbb{R}$ that satisfies

$$2f(x-y) = f(a-y)f(x) + f(a-x)f(y)$$

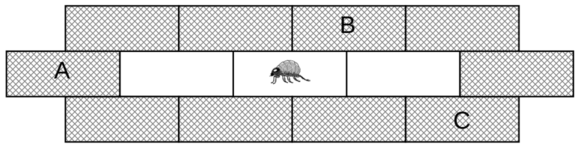
for all $x, y \in \mathbb{R}$.

Solution 3

Put $x =-$

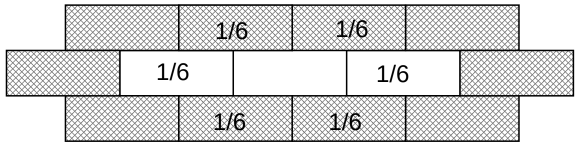
hops onto one of the shaded bricks it is promptly squashed. Otherwise it hops again to one of the neighbouring bricks with equal probability. The flea continues hopping in this fashion and is allowed to visit the same white brick more than once.

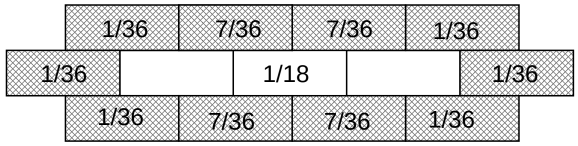
What is the probability p_A that the flea is squashed on the shaded brick labelled A? What is the probability p_B that the flea is squashed on the shaded brick labelled B? What is the probability p_C that the flea is squashed on the shaded brick labelled C?



Solution 4

It is convenient to number the white bricks $1, 2, 3$ from left to right, and let E_n denote





We can now deduce that after many hops

$$p_C = \frac{1}{36} + \left(\frac{1}{18}\right)\frac{1}{36} + \left(\frac{1}{18}\right)^2 \frac{1}{36} + \left(\frac{1}{18}\right)^3 \frac{1}{36} + \dots$$

and by summing the geometric series

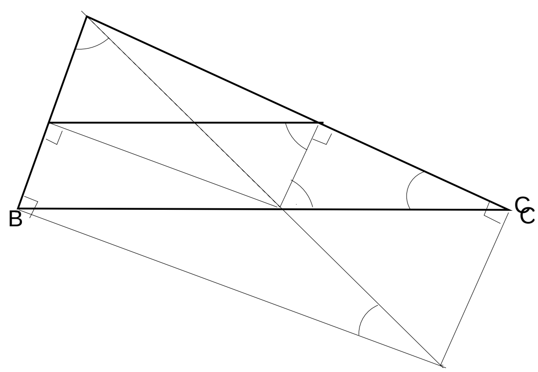
$$p_C = \frac{1}{36}\left(\frac{1}{1-\frac{1}{18}}\right) = \frac{1}{34}.$$

Similarly we find $p_A = \frac{1}{34}$ **and** $p_B = \frac{7}{34}$**.**

Problem 5

Consider a sequence of integers $a_1, a_2, a_3, \dots$ **with** $a_n = a_{n-1} - a_{n-2}$ **if** $n \geq 3$**. Find the sum of the first 2009 terms in the sequence given that the sum of the first 2007 termn**

Now note that $2007 = 3 + 6(334), 2008 = 4 + 6(334), 200 \quad = 5 + 6(334)$ **hence** $s_{2007} = 2a_2 = 2008$ **and** $s_{2008} = 2a_2 - a_1 = 2007$**, from which we deduce** $a_2 = 1004$**,** $a_1 = 1$ **and then** $s_{2009} = a_2 - a_1$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