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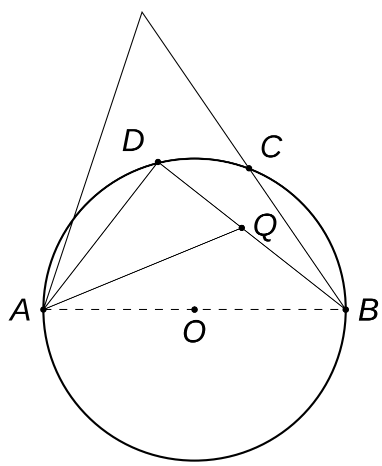
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 25, 2018

1. Suppose that PB intersects the circle at C, as shown.



Once again, $\angle ADB = 90^\circ$, and by the exterior angle theorem, $\angle AQB = \angle ADB +$

2. (a) Note: this question mshould have read: "Explain why, if $a^2 + b^2$ has a xed value, ab is greatest when $a = b$."

For all real numbers a and b,

$$0 \quad (a \quad b)^2$$
$$0 \quad a^2 \quad 2ab + b^2$$
$$ab \quad \frac{a^2 + b^2}{2}$$

Furthermore, equality only holds if $a = b$.

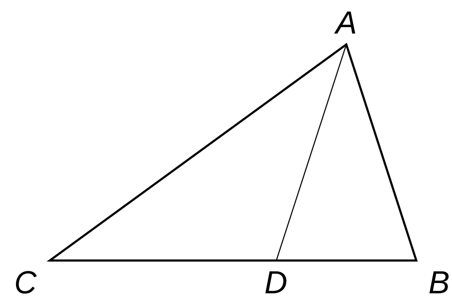
(b)

$$\begin{aligned} x^4 + y^4 &= x^4 + 2x^2y^2 + y^4 \quad 2x^2y^2 \\ &= (x^2 + y^2)^2 \quad 2x^2y^2 \\ &= c^4 \quad 2(xy)^2 \end{aligned}$$

This quantity is minimised when xy is maximised. That is, when $x = y$. If $x = y$, then $x^2 = c^2{=}2$, in which case

$$x^4 + y^4 = c^4 \quad 2 \quad \frac{c^2}{2}^{2} = \frac{c^4}{2}$$

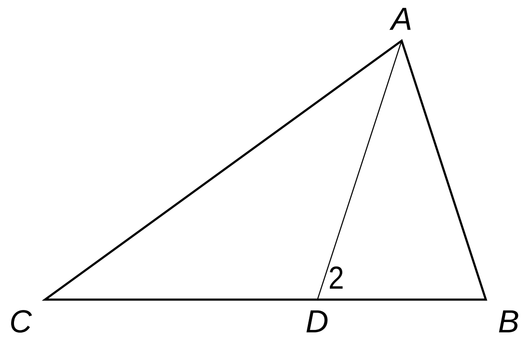
3. Suppose that ABC is a triangle. AD is an angle bisector, and $4ADB$ and $4ADC$ are both isosceles. Let $\backslash CAD =$. Then $\backslash DAB =$. Consider $4ACD$, which is isosceles. There are three possibilities:



(a) $\backslash ACD$ is the vertex;
(b) $\backslash ADC$ is the vertex; or
(c) $\backslash CAD$ is the vertex.

If $\backslash ACD$ is the vertex then $\backslash CAD = \backslash ACD =$ and thus $\backslash ACD = 180^\circ$.

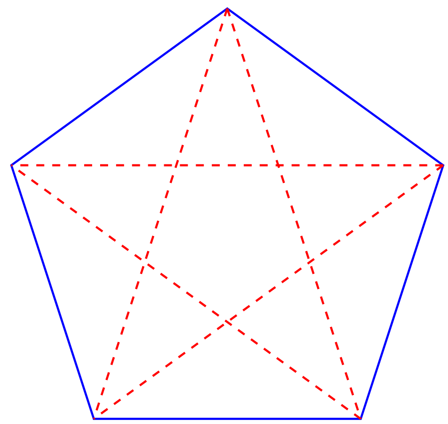
If $\backslash ADC$ is the vertex then $\backslash ACD = \backslash DAC =$ and thus $\backslash ADB = \backslash ACD + \backslash CAD$, since it is the external angle of $\backslash ADC$. So $\backslash ADB = 2$. Since $4ADB$ is also isosceles, it also has a pair of equal angles, and so $\backslash ABD =$ or $\backslash ABD = 2$.



If $\backslash ABD =$, then by the angle sum of $4ABD$, we have $4 = 180°$, so $= 45°$. Hence $\backslash A = 90°$, $\backslash B = \backslash C = 45°$. If, however, $\backslash ABD = 2$, then by the angle sum of $4ABD$, we have $5 = 180°$, so $= 36°$. Hence $\backslash A = \backslash B = 72°$, $\backslash C = 36°$.

If $\backslash CAD$ is the vertex, than we have two cases that are essentially the same as those above, only with the orientations of the smaller triangles reversed. Hence we get the same two solutions.

4.4.



But $b = \dfrac{x}{\cos}$ and $a = \dfrac{c \quad x}{\cos 2}$, so

$$\frac{x}{\cos} = \frac{2(c \quad x)\cos}{\cos 2}$$

Collecting x's and 's to opposite sides of the equation, we obtain

$$\frac{x}{2(c \quad x)} = \frac{\cos^2}{\cos 2}$$

$$= \frac{\cos^2}{2\cos^2 \quad 1}$$

As $\;\; ! \; 0$, $\dfrac{\cos^2}{2\cos^2 \quad 1} \; ! \; 1$, and so $x \; ! \; \dfrac{2c}{3}$.