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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 1, May 7, 2018

1. There are 5 odd integer digits to choose from. Once one is chosen for the first digit, only 4 remain, then 3. So there are $5 \times 4 \times 3 = 60$ 3-digit numbers with distinct odd digits.

2. Use the divisibility rules for 9 and 11 to set up a system of simultaneous equations. $A = 4$ and $B = 6$.

3. The radius of the circumcircle is $\sqrt{a^2 + b^2}$. The sum of the area of the 4 crescents, is

$$\quad \frac{\quad}{2}{}^2 + ab \quad \left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$$= ab + \quad \frac{a^2}{4} + \frac{b^2}{4} \quad \frac{a^2}{4} \quad \frac{b^2}{4}$$

$$= ab$$

4. If either x or y is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one of x^2 or y^2 ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6 and so $x^2 + xy + y^2$ is not a multiple of 10. So the only possibility is that x and y are both multiples of 10, meaning $x^2 + xy + y^2$ is a multiple of 100.

5. (a) $ABCB'$ is a parallelogram since BC is parallel to AB' and CB' is parallel to AB. Similarly $CBC'A$ is a parallelogram. So now we know that A is the midpoint of $B'C'$. Now $\angle B'AC = \angle ACB$ because they are alternate. If D is the point at which the altitude from A meets BC then $\angle DAC = 90^\circ - \angle ACD = 90^\circ - \angle ACB$ so $\angle DAC + \angle B'AC = 90^\circ$, and AD is the perpendicular bisector of $B'C'$.

   (b) Since all the altitudes are also perpendicular bisectors of a triangle, and perpendicular bisectors of a triangle are concurrent, these altitudes are also.

Senior Questions

1. Use polynomial long division to show that

$$\frac{x^4(x \quad 1)^4}{x^2+1} = x^6 \quad 4x^5 + 5x^4 \quad 4x^2 + 4 \quad \frac{4}{x^2+1}:$$

Use this result to show that $I = \frac{22}{7}$. Since the integrand is positive on $[0;1]$, so is the integral I and the result follows.

2. If $y = g(x)$, then

$$\begin{aligned} x &= y^y \\ &= e^{\ln(y^y)} \\ &= e^{y\ln(y)} \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{dx}{dy} &= (\ln(y)+1)e^{yln(y)} \\ &= (\ln(g(x))+1)x \end{aligned}$$

If $x > 1$, this last expression is not zero, so we can take reciprocals to obtain

$$\frac{dy}{dx} = \frac{1}{(\ln(g(x))+1)x}:$$

3. Consider the function $g(x) = f(x) \quad x$. Then $g(0) = f(0)$. If $f(0) = 0$, then $\quad = 0$. So suppose that $f(0) \neq 0$. Since f maps $[0;1]$ to $[0;1]$, this means that $f(0) > 0$. Similarly, consider $g(1) = f(1) \quad 1$. If $f(1) = 1$ then $\quad = 1$, but if $f(1) \neq 1$, then $g(1) < 0$. Since f is a continuous function, then so is g, and since $g(0)$ and $g(1)$ have opposite signs, g has a zero in $[0;1]$. That is, there is a number $\quad 2\ [0;1]$ such that $f(\quad) = \quad$.

4. Let $Q(x) = P(x) \quad x + 7$, noting that $Q(x)$ has roots 17 and 24. Hence

$$P(x) \quad x + 7 = A(x \quad 17)(x \quad 24):$$

In particular, this means that

$$P(x) \quad x \quad 3 = A(x \quad 17)(x \quad 24) \quad 10:$$

Therefore, $x = n_1; n_2$ satisfy $A(x \quad 17)(x \quad 24) = 10$, where A, $(x \quad 17)$, and $(x \quad 24)$ are integers. This cannot occur if $x \quad 17$ or $x \quad 24$ because the product $(x \quad 17)(x \quad 24)$ will either be too large or not be a divisor of 10. Thus $x = 19$ and $x = 22$ are the only values that allow $(x \quad 17)(x \quad 24)$ to be a factor of 10. Hence the answer is $19 \quad 22 = 418$.