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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 2, May 7, 2016

1. If the first digit of n is $1, 2$ or 9, then there is nothing to prove. If the first digit of n is 3, then the first digit of $3n$ is 9. If the first digit of n is $4, 5$ or 6, then the first digit of $3n$ is 1. If the first digit of n is 7 or 8, then the first digit of $3n$ is 2. This completes the proof, has we have exhausted all possibilities.

2. How many numbers between 100 and 500 that are divisible by 7 but not by 21.

   The first and last numbers divisible by 7 between 100 and 500 are 105 and 497 respectively. Therefore, there are $(497 - 105) \div 7 = 56$ numbers between 100 and 500 that are divisible by 7. Since $21 = 3 \times 7$, the round-down of $1/3$ of the 56 numbers that are divisible by 7 is also divisible by 21. Therefore, the answer is the rounded-up of $56 \times 2/3 = 37.222\ldots$, which is 38.

3. Let ABC and DEF be right-angled triangles, with AF and DC their respective altitudes; see figure below. Point G is the intersection of AC and DF. Point H is such that GH is perpendicular to BC. Given $AF = 6$, $GH = 4$ and $FC = 9$.

   (a) Since ABC is a right-angled triangle, by the Geometric mean theorem (see https://en.wikipedia.org/wiki/Geometric_mean_theorem) one has $|AF|^2 = |BF| \times |FC|$, where the notation $|AF|$ here means the length side AF. Therefore, $|BF| = 6^2/9 = 4$; $|BC| = 13$.

   (b) Note that the triangles ABC, DEF and GFC are similar. Therefore,

   $$\frac{|FH| + |HC|}{|DC|} = \frac{|FH|}{|GH|} \quad \text{and} \quad \frac{|FH| + |HC|}{|AF|} = \frac{|HC|}{|GH|}.$$

   Solving the above system of equations gives $|DC| = 12$. Hence, the area of $AGDEB$ is $\frac{1}{2}(|AF| \times |BC| + |DC| \times |FE| - |GF| \times |HC|) = 171$.

4. Using the method of partial fractions, and then noting that the consecrative terms of

Therefore,

$$\int_0^1 (f(x))^2 dx \quad 2\ _1 + 2\ _2 \quad {}^2_1 + \ _1\ _2 + {}^2_2 = 3\ : \tag{1}$$

In particular, the minimal of $\int_0^1 (f(x))^2 dx$ is attained when the RHS of (1) is minimal. Let $F(\ _1;\ _2) = 2\ _1 + 2\ _2 \quad (\ ^2_1 + \ _1\ _2 + \ ^2_2 = 3)$. To nd the critical points of F, we solve the system

$$\frac{@F}{@\ _1} = \quad 2 + 2\ _2 + \ _2 = 0; \qquad , w\ _2 + 2$$

3. If $a = 0$, then $b = 2$ is the only solution.

   Now if $a > 0$, then $3 \quad 2^a$ is always even, so that b is always odd. Hence $b = 2k + 1$ for some integer k. Hence, $3 \quad 2^a = 4k^2 + 4k = 4(k + 1)k$.

   If $a = 1$, then $3 = 2(k + 1)k$, which has no solutions.

   If $a = 2$, then $3 = (k + 1)k$, again has no solutions.

   If $a > 2$, then $3 \quad 2^{a \quad 2} = (k + 1)k$. Since 2 and 3 are coprime, if k is odd, then $k + 1 = 2^{a \quad 2}$ and $k = 3$. Else if k is even, then $k + 1 = 3$ and $k = 2^{a \quad 2}$. Hence, we