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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 16, September 5, 2016

1. Let $f(x) = (x+1)(x+2)(x+3) \ldots (x+n)$, and denote by S_o and S_e the sum odd and even coe cients of the polynomial $f(x)$, respectively. Then

$$S_o + S_e = f(1) = 2 \quad 3 \quad \ldots \quad n \quad (n+1) = (n+1)!;$$

and

$$S_e \quad S_o = f(\quad 1) = 0.$$

Hence, $S_o = \frac{(n+1)!}{2}$.

2. Let A, B, C, D be the length of the edges, and E, F the length of the diagonals of the polynomial formed by the four points; as shown below.

It is given that

$$A + E + D = A + F + B = B + E + C = C + F + D. \tag{1}$$

By the rst and third terms of (1) we have $A + D = B + C$, and by the second and fourth terms of (1) we ave $A + B = C + D$. Therefore $A = C$ and $B = D$. Furthermore, by the rst and second terms of (1) we have $E + D = F + B$, and by third and fourth terms of (1) we have $B + E = F + D$. Therefore $E = F$. Thus, the polynomial formed must be a rectangle.

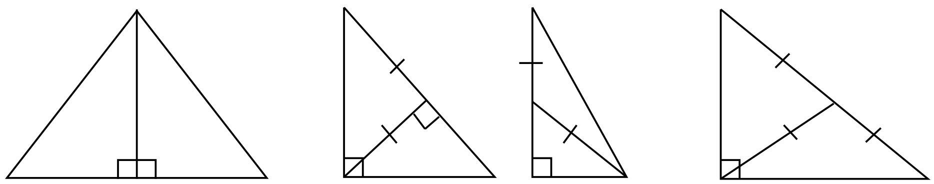
3. We show that

   (a) Any triangle can be dissected into 2 right-angled triangles.

   (b) Any right-angled triangle can be dissected into 1 right-angled triangle, and 1 isosceles triangle.

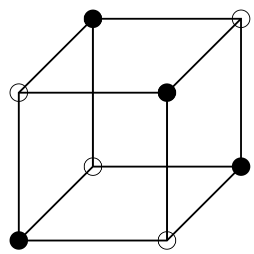
(c) Any right-angled triangle can be dissected into 2 isosceles triangle.

Then given any triangle, we can apply the above three dissecting operation, to form any n 4 isosceles triangles we wish. The following diagrams illustrates the listed constructions



(a) (b) (c)

4. Place 4 black and 4 white dots on the corners of a cube, such that for any black dots the three adjacent corners are white dots, and visa versa for the white dots; as shown below.



If the black dots represent the number 1, and the white dots represents the number 0. Then at each step, all black dots turn into white dots, and all white dots turn into black dots. Hence, after ten steps, the positions of the black and white dots are the same as the initial con guration, but they are not all equal initially.

5. If we set the left turns as   90° rotations, then the right turns are 90° rotations. Since the car is not allow to pass through any place twice, in order for the car to return to its initial location, it must have rotated a net total of 360° or   360°. Hence it has made either 104 or 96 right turns.

6. We rst show that the three hands coincide only at 12 : 00 or 24 : 00. Suppose this occurs again. Consider the angular distance   covered by the hour hand where 0° <   < 360°. The angular distance covered by the minute hand is $360°n +$, where n is the number of revolutions it has made. Since the minute hand moves at 12 times the speed of the hour hand, $360°n +$ = 12 , so that   = $360°\frac{n}{11}$. The angular distance covered by the second hand is $360°m +$, where m is the number of revolutions it has made. Since the second hand moves at 720 times the speed of the hour hand, $360°m +$ = 720 , so that   = $360°\frac{m}{719}$. From $\frac{n}{11} = \frac{m}{719}$, n must be a multiple of 11 and m a multiple of 719 as 11 and 719 are relatively prime. However, this contradicts 0° <   < 360°. This justi es the Steve's claim. If there are two indistinguishable times

Senior Questions

1. By similar triangles it is easy to see that the typical point on the long diagonal of the cube has coordinates $(x; y; z) = (20t; 18t; 15t)$ with $0 \le t \le 1$. We can evaluate the number of unit cubes the diagonal line from $(0; 0; 0)$ to $(20; 18; 15)$ passes through, by considering the number of times the line intersects with the xy, xz and yz faces of each unit cube.

   Let $A; B; C$ represent the event of the line passing through the xy, xz, yz face of any unit cube, respectively. Then the number of times the line passes through the xy face of the unit cubes is $|A| = 15$, because this happens exactly when the z-coordinate of $(20t; 18t; 15t)$ is an integer, which occurs 15 times for $0 \le t < 1$. Similarly, $|B| = 18$ and $|C| = 20$.

   However, we can not just add $|A|$, $|B|$ and $|C|$ to obtain the number of unit cubes the long diagonal passes through, because we would of double counted the event of this line passing thought the edge of the unit cubes; i.e one such event is $A \cap B$ which represents the line passing through the edge of the cubes parallel to the x-axis. We have $|A \cap B| = 3$, since $A \cap B$ occurs when

The sum of the x-coordinates of the other two points of intersections is

$$\frac{b_1 \quad b_3}{a_3 \quad a_1} + \frac{b_2 \quad b_4}{a_3 \quad a_1} = \frac{b_1 + b_2 \quad b_3 \quad b_4}{a_3 \quad a_1};$$

as well.