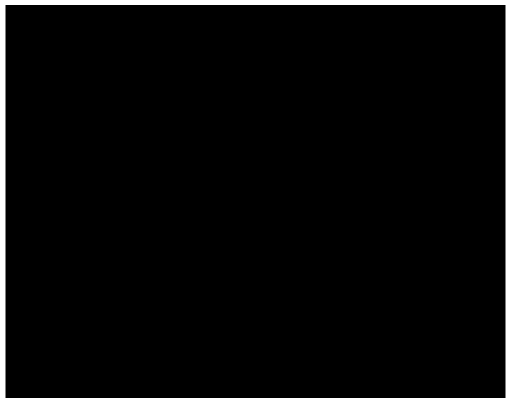
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 3, May 12, 2015[1]

1. Let n be the number of vertices of the polygon. Then there is $n \quad 3$ diagonals connected to each vertex, because the diagonals can not connect a vertex to itself or connect a vertex to two vertices adjacent to it. Also, each diagonal is connected to exactly two vertices. Therefore, the total number of diagonals of a polygon is $n(n \quad 3)=2$. To complete the question, solve the quadratic $n(n \quad 3)=2 = 152$, which gives $n = 16$, $n = \quad 19$. Discard the unrealistic solution $n = \quad 19$.

2. For $17p + 1$ to be a square, there must be some integer9 ution

4. Draw the tangent to the circles C_1 and C_2 at the points T, and let O_3 and O_4 be the points of intersection between this tangent and the lines A_1A_2, B_1B_2 respectively; see below. Then by the tangents to common external point property of circles, we have $jA_1O_3j = jTO_3j = jA_2O_3j$ and $jB_1O_4j = jTO_2j = jB_2O_4j$. So the circle with diameter A_1A_2 has centre O_3 and will pass through the point T. Similarly the circle with diameter B_1B_2 has centre O_4 and will pass through the point T. Since the line passing over O_3 and O_4 is straight, we can conclude that the circle with diameter A_1A_2 and B_1B_2 are tangent to each other at T.



5. Let x, y and z be the total number of apples, peaches and mangoes respectively. If $a_1; a_2; : : : ; a_6$ are the number of apples in each basket, and $p_1; p_2; : : : ; p_6$ are the number of peaches in each basket. Then $p_1 = a_2 + a_3 + a_4 + a_5 + a_6$, $p_2 = a_1 + a_3 + + a_4 + a_5 + a_6$ etc. Therefore,

$$\begin{aligned} y &= p_1 + p_2 + ::: + p_6 \\ &= 0 + a_2 + a_3 + a_4 + a_5 + a_6 + ::: \\ &\quad + a_1 + 0 + a_3 + a_4 + a_5 + a_6 + ::: \\ &\quad + \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad + ::: \\ &= 5(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) = 5x: \end{aligned}$$

We can repeat the above argument for apples and mangoes to get $x = 5z$. In conclusion, the total number of fruit is $x + y + z = x + 5x + 25x = 31x$.

6. Let x be the unique point where the cyclists are allowed to pass each other. Suppose two cyclist have integer speed p and q, with $p > q$, then the faster cyclist is initially in front, and catches up to the slower cyclist at $(q \quad p)=p$ of a lap per one lap the slower cyclist completes; It takes exactly $p=(q \quad p)$ laps (of the slower cyclist) for the two cyclist to overlap, so provided $p=(q \quad p)$ is an integer then the two cyclist will only pass each other at x.

Let M be the least common multiplier of p and q. If $p=(q \quad p)$ is an integer, then so is $p + M=[(p + M) \quad (q + M)]$; If two cyclist only passes each other at x, then by increasing the speed of both cyclist by M, they will still only pass at x. Moreover, $M=[(q + M) \quad M]$ and $M=[(p + M) \quad M]$ are integers; if another cyclist starts at x and is moving at speed M, then he/she will only be over taken by the cyclists moving at speed $p + M$ and speed $q + M$ at the point x.

Can you use the above information to construct 33 cyclist that will only pass at the point x?

Senior Questions

1. After cutting the cheese twice, the pieces of cheese would be in the ratios $1/(1 + x)$, $x/(1 + x)^2$ and $x^2(1 + x^2)$. To divide the pieces of cheese such that there is an equal amount in each pile, we solve

$$\frac{1}{1+x} + \frac{x}{(x+1)^2} = \frac{x^2}{(x+1)^2},$$

which has solutions $x = 1 \pm \sqrt{2}$. Since Alex can pick any number $x > 1$, he can divide the piles in to equal amounts in just two cuts by taking $x = 1 + \sqrt{2}$.

Suppose Alex wants to choose an integer ratio to cut the cheese. Would the above solution still work? if not, then can he keep cutting the cheese into more pieces to eventually have two piles of equal amount?

2. Let $f(x) = ax^b$, then $f'(x) = abx^{(b-1)}$. We want to find a and b such that $f'(f(x)) = a^2bx^{b(b-1)} = x$; i.e $b(b-1) = 1$ and $a^2b = 1$. Solving to get $b = (1 \pm \sqrt{5})/2$ and $a = \sqrt{2/(1 \pm \sqrt{5})}$.

3. (a) Let F_n be the n^{th} term of the sequence, then $F_{n+2} = F_{n+1} + F_n$; the next 3 terms of the sequence are $21, 34, 55$.

   (b)

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
&= F_n + F_{n-1} + F_{n-1} + F_{n-2} \\
&= F_n + F_{n-1} + F_{n-2} + F_{n-2} + F_{n-3} \\
&= \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
&= F_n + F_{n-1} + \ldots + F_2 + F_2 + F_1 \\
&= S_n + F_2 = S_n + 1
\end{aligned}$$