MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, August 25, 2015[1]

1. Write $2^{32} + 2^{17} + 1 = (2^{16} + 1)^2$, and observe that $2^{16} + 1$ is the largest Fermat prime number.

2. First we show that the 5^{th} power of any integer have the same last digit as the original number. For a n-digit long integer x, we can write it as $x = a_1 + 10 \cdot a_2 + \ldots + 10^{n-1} \cdot a_n$, where $a_1, a_2, \ldots, a_n$ are nonnegative integers less than 10. Then

$$x^2 = a_1^2 + 10 \cdot a_1 \cdot a_2 + 10^2 \cdot a_2^2;$$

in words, the last digit of x^2 will be the same as the last digit of a_1^2. Repeating this calculation, we can show that the last digit of x^5 will be the same as the last digit of a_1^5. Now, we can easily verify that a_1^5 has the same last digit as a_1 for $a_1 = 1, 2, \ldots 9$, hence x^5 will have the same last digit as x.

Therefore, the last digit of $1^5 + 2^5 + \ldots + 123^5$ is equal to the last digit of $1 + 2 + \ldots 123$; which is 3.

3.

4. Suppose a is n-digits long, then $b = a(10^n + 1)$. Also, $b = ka^2$ for some integer k. Therefore,

$$k = \frac{b}{a^2} = \frac{10^n + 1}{a}.$$

Since $10^n + 1$ is a $n + 1$ digits long number, the fraction on the RHS of the last equation must be greater than 0 and less than 10. Therefore, the integer $1 < k < 9$. Since $ak = 10^n + 1$, k must be odd. k can not be 1, otherwise a will be $n + 1$ digits long. k can not be $3, 5$ or 9, because $10^n + 1$ is never divisible by those numbers. Thus the only possibility if 7.

5. (a) If we draw a horizontal line across any one of the 1×1 grid squares (we think of the dimension as *vertical* $\times$ *horizontal*), then no matter how the 10×12 paper is folded along the grid lines, this horizontal line will still be horizontal. Similarly, any vertical line will be preserved under the action of folding.

[1] Some problems from UNSW's publication Parabola, and the Tournament of Towns in Toronto.

Thus, if we were to cut horizontally across the thick 1 1 square folded paper, then we are cutting horizontally across each of 1 1 grid squares of the unfold paper. This means we will produce 10 + 1 strips of papers. Similarly, if we cut vertically then we will produce 12 + 1 strips.

(b) Label the rst row along the grid points (i.e the corner of the 1 1 squares) with A's and B's in a alternating fashion. Then label the second row with C's and D's in an alternating fashion. Repeat the AB labelling on the third row, and CD on the fourth row etc, in an alternation fashion until each grid point has been labelled. Then no matter how the 10 12 paper is folded, the $A; B; C$ and D corners will always be folded onto itself.

Thus, if we were to cut the corner labelled by A of the thick 1 1 square folded paper, then we would be cutting out each of the grid labelled with an A in the original unfold 10 12 paper. Since there are 5 6 grid points labelled with an A, it follows that there are 30 + 1 separated pieces due to this cut. We can use a similar argument to work the number of components we get by cutting the other corners of the folded 1 1 paper.

6. Let $a = 10^b$, then we can rewrite the inequality $10 < a^x < 100$ as $1 < bx < 2$. Similarly, if $100 < a^x < 1000$, then $2 < bx < 3$. Suppont 6alw with an

2.