MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 8, June 24, 2014 [1]

1. To count the number of possible passwords for website B, we simply need to look a the number of ways of arranging n symbols when one can choose from 62 symbols (26 lower case letters, 26 upper case letters and the 10 numbers 0-9). To count these, the first symbol has 62 options, the second also has 62, the third has 62 and so on. So for a n digit password, there are 62^n possibilities. Seeing as we can have up to 6 symbols, the total number is

$$\#(B) = 62^6 + 62^5 + 62^4 + 62^3 + 62^2 + 62 = 57\,731\,386\,986$$

To count the number of possible passwords for website A we shall count the number of 6, 7 and 8 symbol passwords separately. The 6 symbol passwords must have exactly 3 letters and 3 numbers. So there are 26^3 possiblities for the letters and 10^3 possibilities for the numbers. Now we must choose how we order the two classes of symbols: there are 6 possible spots to put the 3 letters so there are $\binom{6}{3} = 20$ possible ways of choosing spots which will have a letter and the rest must necessarily have a number. So the number of 6 symbol passwords for website A is

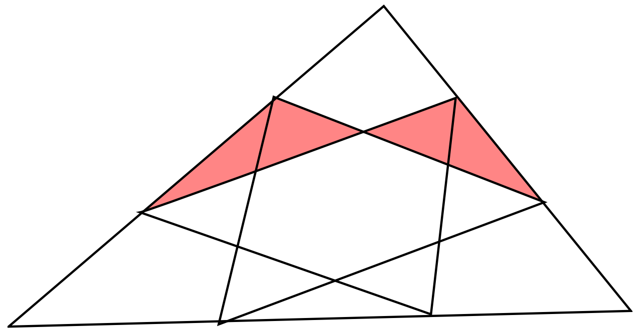
$$\#(A; 3; 3) = 26^3 10^3 \binom{6}{3}$$

where $\#(A; n; m)$ is the number of passwords with n letters and m numbers. We can write out a formula for this as

$$\#(A; n; m) = 26^n 10^m \binom{n+m}{n} \binom{m}{m} = 26^n 10^m \binom{n+m}{n} :$$

So website A is more secure (has more possible passwords).

Going beyond the actual question though, we might ask is this because of website A's passwords are more complicated or merely longer? In fact, let's write $\#_B(n)$ and $\#_A(n)$ as the number of n-symbol passwords for websites B and A respectively, then



4. First, the only way to form a hexagon is as in the picture. It can be shown that the shaded triangles are all equal area (using either congruence or because they stand on equal bases). They each have area $\frac{1}{27}$. The kite-like quadrilaterals at each vertex can be shown to have area $\frac{4}{27}$, making use of the fact that medians cut the area of triangles in half (which triangles and which medians are left to you).

   In all, the area of the hexagon is given by $1 \quad 3 \quad \frac{4}{27} \quad 9 \quad \frac{1}{27} = \frac{2}{9}$.

5. In short, ${}^{9}8 > {}^{8}9$. To see this, if m and n are positive integers with $m > 2n$ then $8^m > 2 \cdot 9^n$. So $8^8 > 2 \cdot 9 > 9$, so ${}^{3}8 > 2 \cdot {}^{2}9 > {}^{2}9$ and then ${}^{4}8 > 2 \cdot {}^{3}9$ and so on. Finally we'd show that ${}^{9}8 > 2 \cdot {}^{8}9$.

6. We can index each number in $\{1, 2, \ldots,$

Senior Questions

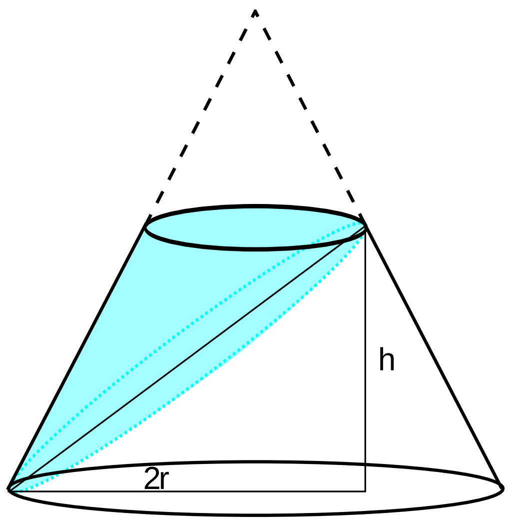
1. If you think of the `disc method' for computing volumes of revolution, we approximate a cone by a pile of thin cylindars stacked on top of one another with radii given by the cone. An oblique cone is just a right cone that's been pushed over, which is like sliding the cylindars so they no longer sit directly on top of one another, but are centred at the, now oblique, centre line. In more formal terms, this means the integral we compute for the volume of a cone

$$\int r(y)^2\,dy = \int_0^h \left(\frac{R_1}{h}y + R_1\right)^2 dy$$

is the same integral we would compute for the volume of an oblique cone.

2. As you can see in the picture, the water makes a truncated oblique cone with elliptical base. By extending the frustrum up to complete the cone we can compute this volume as

$$V_{water} = \frac{1}{3}b_{water}\,h_{water\ oblique\ cone} \quad \frac{1}{3}(\ r^2)(h_{extended\ cone} \quad h)$$



To nd the proportion you want to divide by the volume of the cup which can be calculated by

$$V_{cup} = \frac{1}{3}(\ R^2)h_{extended\ cone} \quad \frac{1}{3}(\ r^2)(h_{extended\ cone} \quad h)$$

To nd the area of the surface of the water, and the height of the oblique cone you'll need to use some 3d geometry. Nevertheless the ellipse has major axis $\sqrt{(r + R)^2 + h^2}$ and minor axis $2\sqrt{Rr}$. The height of the oblique cone is given by

$$\frac{2rRh}{(R \quad r)\sqrt{h^2 + (R + r)^2}}:$$

All in all the proportional volume of the water is $\frac{1}{1+\left(\frac{R}{r}\right)^{\frac{3}{2}}}$.