MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 2, May 13, 2014 [1]

1. (a) To begin, $n^5 - 5n^3 + 4n$ can be factored to $(n+2)(n+1)n(n-1)(n-2)$. You can make the argument here that one of each of these factors must be divisible by one of each of 5,4,3 and 2, and so must be divisible by 120. I thought it was nice to re-write these factors as

$$\frac{(n+2)!}{(n-3)!} = 5!\binom{n+2}{5}$$

and seeing as $5! = 120$, it must be divisible by 120. Note also that for $n = 0; 1; 2$, $n^5 - 5n^3 + 4n = 0$ which is divisible by every number.

(b) Let's write $n^2 + n + 2 = (n+4)^2 - 7(n+2)$ and suppose that it is divisible by 49. If this is the case, then both $(n+4)^2$ and $7(n+2)$ must be divisible by 49, or both $(n+4)$ and $(n+2)$ by 7. This is not possible.

2. If A was truthful about B coming second, then B must be lying about A coming second and C about B coming third, so the order would be ABC.

If A was truthful about C coming first, then B must be lying that C was third and C about A coming first, so the order would be CAB. Either way A beat B.

3. If a number is written in its prime factorisation $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ then for it to be powerful each of the $m_i \geq 2$ and for it to be a perfect power all $m_i = c$, a constant. Thus for n to be powerful but not a perfect power all the m_i must be greater than 2, but not all the same. The smallest then, would be $2^3 \cdot 3^2 = 72$.

4. Draw in the lines shown in red (the diagonals of the square) and the line shown in blue (parallel to $B'D'$ through O), and label the respective points on the blue line A'', B'', C'' and D''. It can be shown (but I'll leave it out here) that the angles drawn in green

all neighbours. So at most there are $9 \quad 10^9=10$ numbers in the collection of ten digit numbers of which none are neighbours.

So we know that at a maximum there are $9 \quad 10^8$ numbers in our collection, and we have an example of a sub-collection that has this many, so the largest possible collection contains $9 \quad 10^8$ numbers!