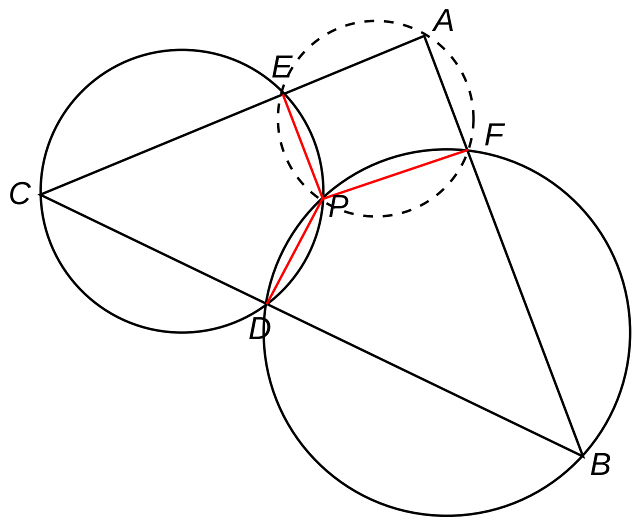
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 1, May 6, 2014

1. Consider, for instance $2(2x + 3y) + (7x + 5y) = 11x + 11y = 11(x + y)$, which is clearly a multiple of 11. Suppose now that $(2x + 3y)$ is a multiple of 11, then so is $2(2x + 3y)$, and if we wish to add a number to this and still get a multiple of 11, we must add a multiple of 11. So $7x + 5y$ must also be a multiple of 11.

2. The number of ways to paints 6 things with 6 colours is 6!. However, a cube may be rotated, so we must divide out the number of orientations a cube has to ensure we are not counting the same colouring twice. Let's x a face as the \top" face { we can then rotate the cube 4 times, keeping the \top" still. A cube has 6 possible faces to choose as the \top", so there are $6 \quad 4 = 24$ orientations of the cube. Thus the number of colourings is $6!{=}24 = 30$.

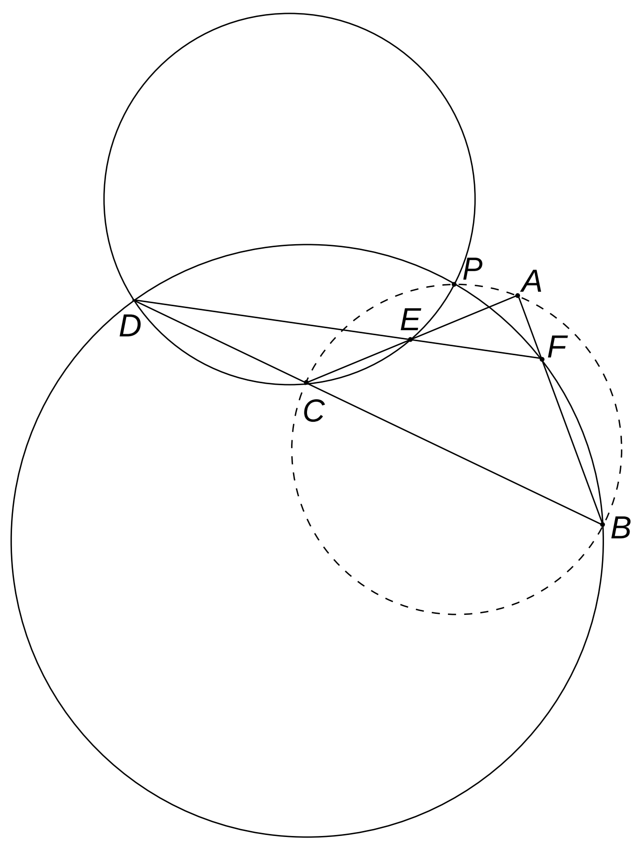
3. (a) $0{:}75_{10} = 0{:}11_2$ ($0{:}11_2 = 1 \quad \frac{1}{2^1} + 1 \quad \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4}$).

   (b)



5. (a) Draw in the red lines DP, FP and EP. Note that the quadrilaterals $EPDC$ and $FPDB$ are cyclic. So $\backslash DPF =$ $\backslash DBF$ and $\backslash EPD =$ $\backslash ECD$. Since the three red lines all meet at a point $\backslash EPF + \backslash EPD + \backslash DPF = 2$ so $2 \ + \backslash EPF = 2 \ + \backslash ECD + \backslash FBD$ so $\backslash EPF = \backslash ECD + \backslash FBD =$ $\backslash EAF$ (angle sum of triangle ABC) and so $AEPF$ is a cyclic quadrilateral so a circle must pass through those four points.

   (b) This solution provided by Michelle Royters.



Draw in the red lines DP, PB and PC. We will show that $\backslash CPB = \backslash CAB$

which proves that $CPAB$ is a circle. Let $\angle DPC = x$ and $\angle CPB = y$. Since angles $\angle DPC$ and $\angle DEC$ stand on the same arc of circle $DPEC$ they are equal, so $\angle DEC = x$. Similarly angles $\angle DPB$ and $\angle DFB$ stand on the same arc of circle $DPFB$ and so are equal, i.e. $\angle DFB = x + y$. Now $\angle DFB$ is exterior to triangle AEF, so $\angle DFB = x + y = \angle AEF + \angle EAF$ and $\angle AEF = x$ as it is vertically opposite $\angle DEC = x$. Thus $\angle EAF = \angle CAB = y = \angle CPB$.

6. A two digit narcisstic number with digits ab must satisfy

$$a^2 + b^2 = 10a + b$$

or

$$a^2 - 10a = b - b^2.$$

Now given an a, we can compute what b must be. For instance, for $a = 1$ we must have $b^2 - b - 9 = 0$ or $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{37}$. Neither of those are the integers $0, 1, \ldots, 9$ so there are no narcissistic numbers of the form $1b$. If $a = 2$ we must have $b^2 - b - 16$ or $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{65}$ which are, again, not integers, so there are no narcissistic numbers of the form $2b$. Repeat for $a = 3, 4, \ldots, 9$ and you'll see that there are no 2-digit narcissistic numbers.

Senior Questions

other $d_j = 0$, so 10^n. But we have already shown that for large enough n, $n9^n$ is not even as great as $10^{n \ 1}$ and so can't possibly be larger than 10^n, meaning the largest possible value of the left hand side above is smaller than the smallest possible value of the right hand side, and so one could not possible equal the other, for large enough n. This all amounts to there being a largest narcissistic number, above which no number can be narcissistic. Since there are only nitely many positive integers smaller than this largest narcissistic number, there can be only nitely many narcissistic numbers in total.