MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 6, June 11, 2013

1. The prime factorisation of $770 = 2 \times 5 \times 7 \times 11$, so assuming by adults we mean over 18 year olds, our two people are 22 and 35.

2. (Disclaimer: Introduction `group theory' answer - this question can be answered more simply by deductive logic, or guess and check (maximum 13 guesses), but this question's close ties to group theory I think warrants a bit of abstract algebra. If you just want the answer, skip to the end ☺)

   Let's write the card shuffler as a function σ, where $\sigma(n)$ is the new position of the nth card after one shuffle. We'll also write iterated shuffles as σ^m, meaning m compositions of the shuffling function σ. As a final piece of notation, we'll introduce `k-cycles', which are written as a collection of numbers in a pair of brackets and indicate that the value of each number is that to its immediate right (or the first position if at the end of the cycle), e.g. (1 2 3) means $1 \to 2$, $2 \to 3$ and $3 \to 1$.

   The information given tells us

   $$\sigma^2 = (1\ 12\ 5\ 2\ 7\ 9\ 11\ 10\ 4\ 13\ 3\ 8\ 6)$$

   We can multiply (compose) cycles together just by tracing from left (i.e. applying the cycles to each number left to right), for example

   $$(1\ 2\ 3)(2\ 1\ 4) = (1)(2\ 3\ 4) = (2\ 3\ 4)$$

   since $1 \to 2 \to 1$, $2 \to 3$, $3 \to 1 \to 4$ and $4 \to 2$.[2] In this manner we can repeatedly multiply σ^2 and we find

   $$\sigma^{26} = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)\ ;$$

   i.e. shuffling 26 times puts the cards back in to the order they originally were. This means the `order' of $\sigma$ is 26, where the `order' of a permutation is how many times

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.
[2] An interesting result is that every permutation can be written as a product of 2-cycles, e.g. (1 2 3) = (1 3)(3 2), and even though this 2-cycle representation is not unique, it is always made up of either an odd or even number of 2-cycles.

you multiply it by itself to get the identify function - one that leaves everything alone like the one above.

Since is, at most, a 13-cycle its order is 13. So the order of could be $1, 2, 13$ or 26 in order to satisfy $^{26} = ()$, but it can't be 26, it's not 1 or 2 from the given information, so it must have order 13.

So now we work out ^{12}, then we can determine so that $^{12} = ()$. I worked out ^{12} by rst performing

$$^2\ ^2 = \ ^2 = (1\ 5\ 7\ 11\ 4\ 3\ 6\ 12\ 2\ 9\ 10\ 13\ 8)$$

then

$$^8 = \ ^4\ ^4 = (1\ 7\ 4\ 6\ 2\ 10\ 8\ 5\ 11\ 3\ 12\ 9\ 13)$$

and nally

$$^{12} = \ ^8\ ^4 = (1\ 11\ 6\ 9\ 8\ 7\ 3\ 2\ 13\ 5\ 4\ 12\ 10)$$

To nd I then wrote it as a 2-cycle representation

$$= (a\ 1)(b\ 2)(c\ 3)(d\ 4) \quad (m\ 13)$$

and work through, from left to right, making sure I put the numbers back where they started. For instance $^{12}(1) = 11$, so set $a = 11$, $^{12}(2) = 13$, so $b = 13$, $^{12}(3) = 2$ so $c = 13$ (I've already made $b = 13$, and so far $2!$ 13 so now I make 13 3 after, so that overall $2 !$ 13). Continuing, we nd

$$= (11\ 1)(13\ 2)(13\ 3)(12\ 4)(12\ 5)(9\ 6)(13\ 7)(13\ 8)(13\ 9)(11\ 10)(11\ 12)(11\ 13)$$
$$= (1\ 10\ 12\ 4\ 5\ 13\ 2\ 3\ 7\ 8\ 9\ 6\ 11)$$

Finally, this means the cards originally ordered $A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K$ become, after one shu e, $J; K; 2; Q; 4; 9; 3; 7; 8; A; 6; 10; 5$.

3. (a) Draw the right angled triangle ABC with right angle at C. Let D be the midpoint of AB, and E a point on AC such that AC ? DE. Then ADE is similar to ABC (three angles equal). Since $AD = \frac{1}{2}AB$ then $AE = \frac{1}{2}AC$ or rather $AE = EC$. Now AED is congruent to CED (two sides equal, $AE = EC$, DE common, and an included angle $\backslash AED = \backslash DEC$). Thus $\frac{1}{2}AB = AD = DC$.

   (b) From part i) we see $DB_1 = B_1C$ and $DC_1 = C_1B$. Note that CB_1A_1 is similar to CAB (two sides in ratio and an included angle). The sides are in ratio 1 : 2 so $A_1B_1 = \frac{1}{2}AB = C_1B$, and so $A_1B_1 = DC_1$. Similarly BC_1A_1 is similar to BAC, so $C_1A_1 = B_1C = B_1A_1$. Thus B_1C_1D and $B_1C_1A_1$ are congruent because they have 3 equal sides.

4. Following the hint, we must have $3m \quad 1 = n$ or $3m \quad 1 = 2n$, since $3m \quad 1 < 3n$. So

$$3(3m \quad 1) \quad 1 = km; \quad k\ 2\ Z$$
$$(9 \quad k)m = 4$$
$$m = \frac{4}{9 \quad k}$$
$$m = 4; 2; \text{ or } 1;$$

or

$$3\frac{3m \quad 1}{2} \quad 1 = km$$
$$9m \quad 3 \quad 2 = 2km$$
$$m = \frac{5}{9 \quad 2k}$$
$$m = 5; \text{ or } 1:$$

Thus the pairs are $(1;1)$, $(1;2)$, $(2;5)$, $(4;11)$ and $(5;7)$.

5. (a) $(12) = 4$, $(30) = 8$

   (b) We can think of (n) as being the number of numbers less than n which are not a multiple of a factor of n (except the factor 1). So if p is prime, its only factors are 1 and p, so every other number is not a multiple of a factor that isn't 1, except p itself. Thus $(p) = p \quad 1$.

   For p^2, the factors are $1, p$ and p^2, so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; p^2$, of which there are p. So $(p^2) = p^2 \quad p$.

   For p^3, the factors are $1, p, p^2$ and p^3, so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; p^2; (p+1)p; \dots; 2p^2; (2p+1)p; \dots$, that is, the multiples of p^2 are contained in the multiples of p, of which there are p^2. So $(p^3) = p^3 \quad p^2$.

   (c) Using the same method as above, the factors of pq are $1; p; q$ and pq, so the multiples of the factors that aren't 1 are $p; 2p; 3p; \dots; qp$ (q of them) and $q; 2q; 3q; \dots; pq$ (p of them), but we don't want to count pq twice. So $(pq) = pq \quad q \quad (p \quad 1)$.

6. We use the fact that the medians divide ABC into 2 equal area pieces, and that S is $\frac{2}{3}$ along the median from A (you can prove these by considering the areas of smaller triangles with the same heights).

   Let the median from A meet BC at P, since ST is parallel to BC triangles APC and AST are similar - 3 angles equal. Since $AS = \frac{2}{3}AP$ then the area of AST is $\frac{4}{9}$ the area of APC which is half the area of ABC so the area of AST is $\frac{2}{9}$ the area of ABC.

Senior Questions

1. Let $f(x) = 2x^n \quad nx^2 + 1$, then $f'(x) = 2nx(x^{n \quad 2} \quad 1)$. So f has stationary points at $x = 0$ and $x = 1$ (since $n > 3$ and odd). Taking the second derivative $f''(x) = 2n(n \quad 1)x^{n \quad 2} \quad 2n$, so $f''(0) = \quad 2n < 0$ and $f''(1) = 2n(n \quad 1) \quad 2n = 2n(n \quad 2) > 0$. So $x = 0$ is a local max and $x = 1$ is a local min.

   Finally $f(0) = 1 > 0$ and $f(1) = 3 \quad n < 0$. Since these are the only stationary points, f is monotonic between/outside of them. Since $x = 0$ is a local max, and positive there is one root for $x < 0$, which is unique since f is monotonic decreasing for $x < 0$. Since $f(0) > 0 > f(1)$ and f is monotonic between 0 and 1 there is exactly one root for $0 < x < 1$. Since $x = 1$ is a local min, $f(1) < 0$ and $f(x)$ is monotonic increasing for $x > 1$ there is exactly one root for $x > 1$. Thus, in total, there are 3 roots.

2. Take the log of both sides and the differentiate both sides with respect to x.

$$\log f(x) = x \log\left(1 + \frac{1}{x}\right)$$
$$\frac{f'(x)}{f(x)} = \log\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$

3. Draw the graph of $y = \frac{1}{t}$ for t between 1 and $1 + \frac{1}{x}$ and we see that the area under the curve is larger than the area of the rectangle with base $1 + \frac{1}{x} - 1$ and height $\frac{1}{1+\frac{1}{x}}$, so

$$\int_1^{1+\frac{1}{x}} \frac{1}{t}\, dt = \log\left(1 + \frac{1}{x}\right) > \frac{1}{x}\frac{x}{x+1} = \frac{1}{1+x}.$$

Thus $\frac{f'(x)}{f(x)} > 0$, and since $f(x) > 0$ for all x so is $f'(x)$.