

MATHEMATICS ENRICHMENT CLUB.  
Problem Sheet 12, August 13, 2013

1

1. Solve  $\frac{x + 3y}{2x + 5y} = \frac{4}{7}$ :

2. If  $a; b; c; d; e$  are real numbers such that

$$a + b + c = 1; \quad b + c + d = 2; \quad c + d + e = 3; \quad d + e + a = 4 \text{ and } e + a + b = 5;$$

which of  $a; b; c; d; e$  is the largest?

3. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

4. Take any triangle  $ABC$  and show how to construct an equilateral triangle inside  $ABC$  whose vertices touch the sides of  $ABC$ . (Hint: Start by constructing an equilateral triangle outside  $ABC$  with  $AB$  as one of its sides.)

5. 100 Queens are placed on a 100x100 chessboard so that no two attack each other (Queens attack each other if they are both on the same row, column or diagonal). Prove that each of the four 50x50 corners of the board contains at least one Queen.

6. (a) Verify that

$$\begin{aligned} x^{15} - 1 &= (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1) \\ &= (x^5 - 1)(x^{10} + x^5 + 1): \end{aligned}$$

(b) Hence factor  $x^{15} - 1$  as a product of prime factors.

(c) Can you factorise  $x^{15} + 1$  as a product of prime factors?

---

<sup>1</sup>Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni. Senior problem 1 provided by I. Woodhouse. Some problems are from the Tournament of Towns

Senior Questions

1. Let  $z = \cos \theta + i \sin \theta$ ,  $z \neq 0$ .

(a) Show that for

$$z^n + \frac{1}{z^n} = 2 \cos n\theta;$$

for  $n = 0; 1; 2; \dots$ .

(b) Show that

$$z + \frac{1}{z} = z^{n-1} + \frac{1}{z^{n-1}} = z^{n-2} + \frac{1}{z^{n-2}} = \dots = z^n + \frac{1}{z^n};$$

for  $n = 1; 2; 3; \dots$ .

(c) Hence deduce that

$$\cos(n\theta) = 2 \cos \theta \cos((n-1)\theta) - \cos n\theta;$$

2. How many real roots does the equation  $\cos x = 3(1 - \sin x)$  have? Use Newton's method to find an approximate value of the smallest one and hence find the largest one.